What the electromagnetic vector potential describes

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An explicit physical interpretation of the electromagnetic vector potential is here pointed out—as field momentum available for exchange with kinetic momenta of charged matter. It is shown that the vector potential can be quite as directly measurable, without recourse to only quantum-mechanical effects, as are scalar potential differences and the force fields **E**, **B**. This suggests, in keeping with quantum electrodynamics, that the equations for potentials may be regarded as more "basic" than the Maxwell equations—but only because the potentials most directly represent interaction energy-momenta through which fields and charges become observable.

Any electromagnetic field may be described by giving **E**, **B**($\mathbf{r}$, t) *or* by giving potentials **A**, $\phi(\mathbf{r}, t)$ from which **B**, **E** are derivable, via

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \partial\mathbf{A}/c\partial t. \tag{1}$$

However, only **E**, **B** are usually regarded as "real" physical fields. It remains quite customary to make statements[1] that at least the vector potential **A** has no physical meaning—to regard its introduction as no more than a mathematical convenience, useful as a step in solving the Maxwell equations for **E**, **B**. The author has contradicted such statements to several generations of students, on the basis of some quite elementary findings, and has lately been urged to make these known to a wider, and perhaps more critical, audience.

A reason for hesitation has been doubt that questions about "reality" have any effective meaning for physics; it seems enough that a concept be useful, as is **A** in electromagnetism. That point of view is well expressed by Feynman,[2] who goes on to stress the unavoidable role of **A** in quantum theory. The best-known explicit example of such a role is provided by the Bohm-Aharonov effect,[3] which seems to have led to agreement that the vector potential acquires physical meaning only through its quantum-mechanical effects. The effort here will be to show that **A** has always had a more explicit physical meaning, and direct measurability, already in "classical" situations.

A IN THE POINT-CHARGE EQUATION OF MOTION

It is the "operational" definitions of **E** and **B**, their detectability through forces $q\mathbf{E}$ and $\mathbf{v} \times (q\mathbf{B}/c)$ on a test-charge q, that is supposed to lend them "reality." Their definition thus stems from the equation of motion for a point charge.

$$d(M\mathbf{v})/dt = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}/c], \tag{2}$$

with **E**, **B** to be evaluated at positions $\mathbf{r}_q(t)$ of the point particle. Even when care is taken to evaluate the mass $M(v)$ of the particle "relativistically," this equation is actually valid only at speeds low enough for "retardation" effects in the charge's moving Coulomb (self-) field—with the consequent possibility of freeing radiation—to remain undetectable. This serves to emphasize that it is through their role in "quasistatic" situations—in the limit $\mathbf{v} \to 0$—that the fields become unambiguously identifiable. It is generally conceded sufficient to carry out the primary defining measurements at low speeds and for static fields. Like all physical concepts, the **E** and **B** present in different situations gain their extended meanings through processes of generalization[4]—such as are encompassed in the Maxwell equations.

Operational definitions of ϕ, **A** should now be expected to stem from the equation of motion (2) when it is reexpressed in terms of the field description by the potentials, through substitutions from (1):

$$\frac{d}{dt}[M\mathbf{v} + (q/c)\mathbf{A}] = -\nabla q[\phi - (\mathbf{v}/c)\cdot\mathbf{A}]. \tag{3}$$

This is also the form that follows most directly from the variational principle, and the Lagrangian or Hamiltonian representations of mechanics—all dealing with energy and momentum exchanges without requiring an explicit conception of forces, as for the Newtonian prescription (2). Equation (3) gives changes in the "conjugate" momentum, $\mathbf{p} \equiv M\mathbf{v} + q\mathbf{A}/c$, that are generated wherever there are gradients in an "interaction energy" $q[\phi - \mathbf{v}\cdot\mathbf{A}/c]$. The discrimination and definition of the conjugate momentum is very important to physical theory in general, as is illustrated by the fact that it is **p**, and not $M\mathbf{v}$, that must be given the Schrödinger representation, $+(\hbar/i)\nabla$, in transcriptions to wave mechanics. It is $\Delta\mathbf{p}$, rather than $\Delta(M\mathbf{v})$, that is the subject of the fundamental uncertainty principle when the particle is charged and in an electromagnetic field.

In contrast to treatments of the vector potential, an operational definition of the scalar potential ϕ has always been admitted—as "potential *energy*" per unit charge, "stored" whenever charge is brought into a preexistent field, and equated to the work needed to do this. The connection is usually made explicit by evaluating the work rate (power being supplied) as it follows from the equation of motion (2) or (3):

$$\frac{dT}{dt} = q\,\mathbf{E}\cdot\mathbf{v} \to -\frac{d}{dt}q\phi[\mathbf{r}_q(t)]. \tag{4}$$

Here, T in the particle's kinetic energy and the last equality holds only in a *static* field—the variation with time arising only as the particle moves into new positions in the static field. This is sufficient to consider for defining measurements just as it is taken to be for the definitions of **E** and **B**. Then a conserved energy,[5]

$$H = T + q\phi(\mathbf{r}), \quad \text{a constant,} \tag{5}$$

becomes definable—to describe the balance of the energies

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as these are exchanged between the motion and the "store."

The word "store" may be somewhat more apt than the usual "potential energy" because $q\phi(\mathbf{r})$ actually represents electromagnetic field energy existing as a part of the total field energy present whenever a charge q has been placed at a point $\mathbf{r}$ in an external field. It is the only part that depends on the position of the charge and so is the only part "locally" available for exchange with the particle's kinetic energy during motions changing the position. The basis for such conclusions will be amplified in the next section.

The vector potential is absent from the energy expression (5) primarily because the magnetic force in (2) is always perpendicular to the motional displacements, $\mathbf{v}\,dt$, and so does no work. The magnetic field only redirects momenta, and an interpretation of the vector potential from which it derives must come from the momentum exchanges described in (3).

Just as the scalar component ϕ of the potential field became separately identifiable while the energy (5) was conserved, so the role of the vector potential $\mathbf{A}$ becomes least ambiguous while at least a component[6] of the conjugate momentum, $\mathbf{p} = M\mathbf{v} + q\mathbf{A}/c$, is conserved. This can be arranged by constraining $\mathbf{v}$ to a trajectory on which the gradients on the right side of (3) vanish—as in the explicit example to be discussed below. With $\mathbf{p}$ constant on it, any variation of $\mathbf{A}$ along the trajectory will require $M\mathbf{v}$ to vary in compensation and an exchange of momenta between the kinetic form and a "store" must be taking place. Thus—just as $q\phi$ serves as a "store" of field-energy—so $q\mathbf{A}/c$ measures a "store" of field *momentum* available to the charge's motion. Those who prefer to call $q\phi$ a potential energy might adopt the name "potential momentum" for $q\mathbf{A}/c$.

The interpretation here might have been anticipated as soon as it was learned that—mapped on four-dimensional space-time in conformity to the principle of relativity—$\mathbf{A}$ and ϕ form the four components of a four-vector, interchangeable merely by being viewed from relatively moving frames of reference, as are components of $\mathbf{E}$ and $\mathbf{B}$ (already true in low-velocity "Galilean" relativity!). *Any* energy, like $q\phi$, is also the fourth component of a four-vector and, moreover, this has a vector momentum as its three "spatial" components.

That the interpretation amounts to more than just the adoption of suitable names is suggested when the connections to the usual formulations of field energy and field momentum are investigated—as in the next section.

FIELD MOMENTUM

Any volume $\oint dV(\mathbf{r})$ of electromagnetic field can be treated as a mechanical system possessing energy, momentum, and even mass subject to gravitation. In the gaugings usually adopted, the energy and the vector momentum are each distributed over the volume with the respective densities:

$$w(\mathbf{r},t) = (E^2 + B^2)/8\pi, \quad \mathbf{g}(\mathbf{r},t) = (\mathbf{E} \times \mathbf{B})/4\pi c. \quad (6)$$

The total mass of the field is $(W = \oint dV\,w)/c^2$, being constant if there are no fluxes through the surface enclosing the volume.

The primary interest here is in a system composed of a point charge q at a fixed position $\mathbf{r}_q$ in a static external field $\mathbf{E}_0 = -\nabla\phi$, $\mathbf{B}_0 = \nabla \times \mathbf{A}(\mathbf{r})$. The point charge itself contributes a Coulomb field $\mathbf{E}_q(\mathbf{r} - \mathbf{r}_q)$. Then the energy in the entire field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_q$, $\mathbf{B} = \mathbf{B}_0$—obtained by volume integration of the density $w(\mathbf{r})$, defined as in (6)—may be expressed as a sum of three contributions:

$$W = W_0 + W_q + U(\mathbf{r}_q). \quad (7)$$

Here W_0 is the mass-energy of the external field $\mathbf{E}_0$, $\mathbf{B}_0$ in the absence of q, W_q/c^2 is an invariant Coulomb field-energy contribution to the rest mass of the charged particle, and $U(\mathbf{r}_q)$ arises from the interference (interaction) between the Coulomb and external fields,

$$U(\mathbf{r}_q) = \oint dV(\mathbf{r})\,\frac{\mathbf{E}_0(\mathbf{r}) \cdot \mathbf{E}_q(\mathbf{r} - \mathbf{r}_q)}{4\pi} = -\oint \frac{dV}{4\pi}\,\mathbf{E}_q \cdot \nabla\phi = q\phi(\mathbf{r}_q), \quad (8)$$

after discarding a vanishing surface divergence (or using the "Hermiteanship" of $-i\nabla$) and letting $\nabla\cdot\mathbf{E}_q = 4\pi q\delta(\mathbf{r} - \mathbf{r}_q)$. It is this result that identifies $q\phi$ as an "interaction" part of the electromagnetic field energy, the part that varies with the position $\mathbf{r}_q$ and so $U(\mathbf{r}_q)$ serves as a potential energy available to motions of q.

It might still be remarked that while q has a velocity $\mathbf{v} \equiv \dot{\mathbf{r}}_q(t)$, it generates a $\mathbf{B}_q = \mathbf{v} \times \mathbf{E}_q/c$—with $\mathbf{E}_q$ now a moving Coulomb field. That produces an additional interaction energy density $\mathbf{B}_0\cdot\mathbf{B}_q/4\pi$ which, after appropriate identifications, supplements (8) to the interaction energy:

$$U = q[\phi - \mathbf{v} \cdot \mathbf{A}/c], \quad (9)$$

a result of some interest on two counts. First, it coincides with the form taken by the interaction energy in (3). Second, after $\phi - \mathbf{v}\cdot\mathbf{A}/c$ is "dilated" by the usual factor common to masses and energies as viewed from relatively moving frames, it becomes identical with the scalar potential ϕ', by itself, that exists relative to the instantaneous rest frame of q. That such outcomes are only to be expected emerges with more formal completeness, if not more perspicuously, when the equations of motion are derived[7] in manifestly Lorentz-covariant forms.

Now the volume-integrated resultant, $\oint dV\,\mathbf{g}$, of the field momenta (6), in the entire field $\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_q$, $\mathbf{B}_0$ that led to (7), is to be considered. Again there will be a contribution from the interaction between the external field and $\mathbf{E}_q$, amounting to

$$\mathbf{P}(\mathbf{r}_q) \equiv \oint dV(\mathbf{r})\,\frac{\mathbf{E}_q(\mathbf{r} - \mathbf{r}_q) \times \mathbf{B}_0(\mathbf{r})}{4\pi c}. \quad (10)$$

This is the part of the total field momentum that changes to other (quasistatic equilibrium) values when the position $\mathbf{r}_q$ is changed. The difference must be imparted to the kinetic momentum of the particle in any motion changing $\mathbf{r}_q$—when the particle and the external field are isolated except from each other, thus forming a system which must conserve its total momentum.

The result (10) can be made more explicit by introducing the sources $\mathbf{j}(\mathbf{r}) = (c/4\pi)\,\nabla \times \mathbf{B}_0$ that the magnetostatic field must have, according to Ampere's law, and letting $\mathbf{E}_q = -\nabla\phi_q$ with $\phi_q = q/|\mathbf{r} - \mathbf{r}_q|$. Then

$$\mathbf{P} = \oint \frac{dV}{4\pi c}\,\mathbf{B}_0 \times \nabla\phi_q = \oint \frac{dV}{4\pi c}\,\phi_q \nabla \times \mathbf{B}_0 = \frac{q}{c}\oint \frac{dV\,\mathbf{j}}{c|\mathbf{r} - \mathbf{r}_q|}. \quad (11)$$

The last integral should be recognized as just the vector potential $\mathbf{A}(\mathbf{r}_q)$—in the "solenoidal" gauge usual in statics ($\nabla\cdot\mathbf{A} = 0$)—that arises from the same sources as does $\mathbf{B}_0$, so that $\mathbf{B}_0 = \nabla \times \mathbf{A}$. Thus

$$\mathbf{P}(\mathbf{r}_q) = q\mathbf{A}(\mathbf{r}_q)/c \tag{12}$$

represents the field momentum available to motions of q, as concluded in the preceding section.

The derivation here has called attention to the fact that gauging field momenta as in (6), for static fields, corresponds to using the special divergenceless ("solenoidal") gauge for the vector potential in (12). It is perhaps the wide choices of gauge permitted for representing the vector potential that have led some to deny it physical meaning. However, choices must be made in representing measurements on any continuously variable physical quantity. Changing the gauge of $\mathbf{A}$ has no physical consequences, and the "gauge invariance" of descriptions by potentials is no reason for denying them physical meaning. All the conjugate momenta, Hamiltonians and Lagrangians of mechanics in general permit similarly wide choices of gauge.[4]

As in the example to be presented in the next section, the field momentum (12) may become available at points $\mathbf{r}_q$ completely outside the space of $\mathbf{B}_0$. All that is necessary for the integral (10) not to vanish is that the Coulomb field, of the charge localized at $\mathbf{r}_q$, should overlap on the $\mathbf{B}_0$ distribution.

The last remark indicates that a localization of an "interaction" field energy and/or field momentum outside a field with which the charge is interacting is really owed to the fact that even a point charge is, in a sense, not entirely localizable where only the charge exists. Part of the mass of the charged matter is distributed more widely, wherever its self-field penetrates, and interactions ($\sim\mathbf{E}_q\cdot\mathbf{E}_0/4\pi$ and $\mathbf{E}_q \times \mathbf{B}_0/4\pi c$) exist wherever the self-field overlaps on an external one. The $q\phi$ and $q\mathbf{A}/c$ are *joint* properties of the superposed fields arising from their interference—important because they determine the processes through which fields and charges become observable. Expressions like (6) still provide the best descriptions of a field and its mass-energy distribution when it is *isolated*—but even those descriptions are results of the observing procedures (they are thus derived![7]). The picture here can be completed by adding such facts as that, when $q\phi < 0$, there is a positive "binding energy" of the two entities: point charge q and external field ϕ.

MEASURABILITY OF A

The direct measurability of ϕ (within an additive "gauge constant") is well known—particularly as equal to the kinetic energy gained by a charge in "falling" through a potential difference. Every measurement of $\mathbf{E}$ as a force per unit charge can be reinterpreted in terms of an energy gain from ϕ.

Demonstrating the measurability of $\mathbf{A}$ is plainly most urgent for a situation having a space with $\mathbf{A} \neq 0$, yet $\mathbf{E} = 0$ and $\mathbf{B} = 0$ in it. Such a space exists outside an effectively infinite solenoid carrying some current I in n loops per unit length along a cylinder having some radius $r = a$. Then the $\mathbf{A}$ everywhere has only an azimuthal component, $A = A_\varphi$ in parallel to I, and

$$A(r < a) = 2\pi nIr/c, \quad A(r > a) = 2\pi nIa^2/cr. \tag{13}$$

Whereas inside the solenoid $A(r < a)$ has the well-known curl $B = B_z = 4\pi nI/c$, parallel to the cylindrical axis, the $A(r > a)$ outside has no curl and $B(r > a) \equiv 0$.

The situation here is one suggested by Aharanov and Bohm[3] for testing the existence of a quantum-mechanical effect on de Broglie waves of electrons passing around the solenoid. The wave amplitudes are proportional to $\exp(i\mathbf{p}\cdot\mathbf{r}/\hbar)$ when $\mathbf{p} = M\mathbf{v}$, as in the absence of $\mathbf{A}(\mathbf{r})$, but must be generalized to:

$$\exp[(i/\hbar)\textstyle\int d\mathbf{r}\cdot\mathbf{p}(\mathbf{r})] = \exp[(i/\hbar)\textstyle\int d\mathbf{r}\cdot(M\mathbf{v} + q\mathbf{A}/c)] \tag{14}$$

in the neighborhood of the solenoid [see the remarks following (3)]. Chambers[3] detected the phase shifts $\int d\mathbf{r}\cdot(q\mathbf{A}/c\hbar)$ in a "double-slit" interference experiment, using an equivalent of the solenoid (a permanently magnetized iron "whisker," small enough to fit between slits close enough to each other to produce a distinct interference pattern). Thus a physical meaning of $\mathbf{A}$, as a de Broglie wave "phase-shifter," was established. According to the interpretation of $\mathbf{A}$ proposed here, momentum exchanges between $M\mathbf{v}$ and field momentum $q\mathbf{A}(\mathbf{r})/c$ played the essential role in shifting the phase.

For a purely classical measurement of $A(r > a)$, a "macroscopic" bead bearing a charge q might be placed at the radius $r > a$. According to the discussions above, this cannot be done without adding $q\mathbf{A}(\mathbf{r})/c$ to the field momentum present. The bead can be constrained to a trajectory on which $p = mv + qA(r)/c = p_\varphi$ is conserved, by letting it slide freely on a circular fiber of insulator material that coincides with an $A = A_\varphi$ field line at radius r. The solenoidal A is constant along such a circle, there will be no gradient in the azimuthal component of (3), and indeed

$$dp_\varphi/dt = (d/dt)\,(mv + qA/c) = 0 \tag{15}$$

on the trajectory. With the uniform A, there will also be no momentum exchanges between mv and qA/c to observe—unless the equilibrium being maintained by the "agency" supplying a steady solenoid current I is disturbed. This can be done simply by interrupting the solenoid circuit and letting I and A die out. Then $A(t) \rightarrow 0$ exponentially, with a time constant prolonged by the self-inductance of the solenoid. As for all such "macroscopic" processes, even the exponential decays of $I(t)$ and $A(t)$ can easily be made slow enough for any radiative effects to remain undetectable. A "quasistatic equilibrium" can be assumed at every stage, and this, together with the cylindrical symmetry, leads to a persistence of the relative spatial distribution (13) of the field—proportional to the $I(t)$ throughout the decay. Thus the "adiabatic" constancy (15) of $p = mv(t) + qA(t)/c$ persists, and the value of this constant is just $p = qA(t = 0)/c$ if the bead is started from rest. Then its observable kinetic momentum, $mv(t \rightarrow \infty)$, after $A \rightarrow 0$ is given by

$$p = mv(t \rightarrow \infty) = qA(t = 0)/c. \tag{16}$$

The result is the desired measurement of the (initial) A, as equal to a momentum gain, just as the scalar potential ϕ can be measured by kinetic energy gains.

The kinetic momentum gain in such processes as the above is well known, and the conclusion can be accepted without an actual experimental test. For example, the so-called "betatron principle" relies on the existence of similar momentum gains, but the conventional calculations leave the role of the vector potential unmentioned. Again the "quasistatic equilibrium" is presumed and one speaks of the changing magnetic flux $\pi a^2 B(r < a) = 4\pi^2 nIa^2/c$ within the trajectory $r > a$. This induces a Faraday emf

$$2\pi rE \equiv 2\pi rE_\varphi = -4\pi^2 n\dot{I}(t)a^2/c^2 \qquad (17)$$

along the trajectory—with $E_\varphi > 0$ if $\dot{I}(t) < 0$ as above. The electric force thus produced leads to

$$d(mv)/dt = qE = (q/c)[-2\pi n\dot{I}a^2/cr] \qquad (18)$$

and

$$mv = (q/c)[2\pi nIa^2/cr] \qquad (19)$$

from a start with $v = 0$ and a finish with $I = 0$. It only takes noticing further that the square bracket here is just $A(r > a)$ of (13) and then the long-known result (19) could have always been regarded as a measurement of a vector potential.

The calculation using B values not at the site of the charge treats the field as having "action-at-a-distance," something possible to do only in at least "quasistatic" equilibria, and contradicting tenets on which field theories, particularly relativistic ones, are based. That such momentum gains as (19) arise at the cost of field momentum is generally accepted, but then the question of how, in the above situation, field momentum can be localized where $\mathbf{g} = \mathbf{E} \times \mathbf{B}/4\pi c$ of (6) vanishes must be answered—as done by the findings (10) and (12).

It is also true that a Faraday-induced electric force, $\mathbf{E} = -\partial\mathbf{A}/c\partial t$ of (17) and (1) does appear on the trajectory. However, a magnetic field is also needed there for $\mathbf{E} \times \mathbf{B}/4\pi c \neq 0$. A magnetic field there could only have arisen from a secondary "Maxwell induction," via $\nabla \times \mathbf{B} = \partial\mathbf{E}/c\partial t$, requiring $\ddot{I} \neq 0$ in the slow changes—yet the momentum gains of (18) occur even from uniformly changing currents, with $\ddot{I} = 0$. Even when $\ddot{I} \neq 0$, the Maxwell-induced $\mathbf{B}$ is part of the negligible radiation field. It has the direction parallel to the cylinder axis and helps form a cylindrical wave of radiation, propagating a radially directed momentum that is cancelled by the constraint force keeping the charge on the circular trajectory. That such radiation pressure is completely undetectable in such low-frequency radiation is well known. All "alternating current" theory is based on the negligibility of such radiations.

SOME THEORETICAL IMPLICATIONS

The direct detectability of the potentials ϕ and $\mathbf{A}$, comparable to that of $\mathbf{E}$ and $\mathbf{B}$, makes is possible to regard descriptions of electromagnetic fields by potentials quite as "fundamental" as descriptions by $\mathbf{E}$ and $\mathbf{B}$. Indeed, the potentials might be considered the more "basic," with (1) providing definitions of $\mathbf{E}$ and $\mathbf{B}$ as "derived" concepts. Potentials represent field energies and field momenta, per unit charge, as those participate in the universal conservation of energy and momentum, whereas force and work rate per unit charge, can be regarded as merely convenient terms for the transfer rates.

The equations from which the Lorentz potentials A_ν $(\mathbf{A},\phi)$ arising from given sources j_ν $(\mathbf{j},\rho)$ are derived,

$$\partial_\mu^2 A_\nu = -4\pi j_\nu/c, \quad \partial_\mu A_\mu = 0, \qquad (20)$$

can displace the Maxwell equations at the basis of electromagnetic theory. The Maxwell equations follow from (20) whenever the antisymmetric field tensor $F_{\mu\nu}$ $(\mathbf{E},\mathbf{B}) \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is defined. The equations (20) follow the more directly from the Variational Principle applicable to all fully formulated systems. They follow, as "Euler-Lagrange" equations with constraints, from a Lagrangian density best specified using the A_ν as "generalized coordinates" and $\partial_\mu A_\nu$ as "generalized velocities":

$$\mathcal{L} = -(\partial_\mu A_\nu)^2/8\pi + j_\nu A_\nu/c. \qquad (21)$$

It is true that, because of the gauge invariance characteristic of Langrangians, the "free field" Lagrangian (with $j_\nu \equiv 0$) can be reexpressed in terms of $F_{\mu\nu}$ $(\mathbf{E},\mathbf{B})$. However, integral representations would be needed for the "interaction Lagrangian" $j_\nu A_\nu c$.[8] The variational principle is also applicable to quantum-mechanical descriptions, which only require operator values for A_ν. This accounts for the quite indispensable role[2] of A_ν in quantum electrodynamics.

[1]A forthright one of numerous examples may be found on p. 65 of F. Rohrlich's definitive treatise on *Classical Charged Particles* (Addison-Wesley, Reading, MA, 1965). Some textbooks avoid the issue.

[2]R. P. Feynman, R. B. Leighton, and M. Sands, *The Feynman Lectures* (Addison-Wesley, Palo Alto, CA, 1965), p. 15-14.

[3]Y. Aharanov and D. Bohm, Phys. Rev. **115**, 485 (1959). Its experimental confirmation was first achieved by R. E. Chambers, Phys. Rev. Lett. **5**, 3 (1960).

[4]Note remarks about "continua" of operational definitions on p. 35 in, E. J. Konopinski, *Classical Descriptions of Motion* (Freeman, San Francisco, 1969).

[5]In the general case of nonstatic fields, $H = T + q\phi(\mathbf{r},t)$ is known as an "Hamiltonian" energy—more often "gauged" so as to include the invariant rest energy of the particle, in relativistic formulations. In the latter, $dH/dt = \partial H/\partial t = q\partial\phi/\partial t$ is equivalent to a "fourth component" of a four-vector equation of motion having equivalents of (3) for its "spatial" components in four-dimensional space-time. The conjugate momentum $\mathbf{p} = M\mathbf{v} + q\mathbf{A}/c$ and H/c form the four components of a four-vector just as $M\mathbf{v}$ and $(T + mc^2)/c$ do—and also $\mathbf{A},\phi$.

[6]Vector potentials must be measured a component at a time, just as the magnetic force $\mathbf{v} \times (q\mathbf{B}/c)$ measures only a magnetic field component transverse to $\mathbf{v}$, for a given "test velocity" $\mathbf{v}$.

[7]Such matters are enlarged upon in the author's book, in preparation: Electromagnetic Fields and Relativistic Particles.

[8]Integrated over a point charge, this is just the negative of the interaction energy (9).