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Why does 0.999... = 1?

A perennial question and number sense

Most mathematics teachers, at some point in their careers, will be asked about the equivalence of 0.999... and 1. Most students find it quite counter-intuitive and difficult to believe. At its heart is some profound mathematics, but it is explicable in reasonably simple terms. The first way that it was 'proved' to me is as follows:

Let $x = 0.999...$
$\therefore\ 10x = 9.999...$
Subtracting these equations gives:
$9x = 9$
$\therefore\ x = 1$

This explanation convinces some students but not all. In my experience, those remaining unconvinced include some who are perfectly competent with the algebraic manipulation involved and who can readily follow the steps in the argument. In spite of this, they just do not believe it. This, of course, raises some important questions about the way that these students learned to manipulate algebraic equations. The purpose of this paper is to describe a range of alternative ways that can be used to convince students that 0.999... is indeed 1, and to discuss the ways in which these explanations either rely on students' existing number sense and/or can be used to strengthen their number sense by making connections between the various explanations and the underlying mathematics.

For those who do not 'know' algebra (including students in the middle grades) and equally for those who do not 'believe' their algebra, a most convincing explanation seems to be:

$$\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=1$$

and $\frac{1}{3}=0.333...$

Students can convince themselves of this latter equality either by using a calculator to divide 1 by 3 or by carrying out the pencil and paper division of 1 by 3.

Clearly, $0.333... + 0.333... + 0.333... = 0.999...$
$\therefore\ 0.999... = 1$

In requiring students to accept that 1/3 = 0.333... this explanation relies on similar logic as is required to accept the initial proposition that 0.999... = 1; i.e., that an infinite recurring decimal is equivalent to a fraction. It seems that the fact that students can perform a division to establish the equivalence of 1/3 = 0.333... is the key to its plausibility.

It is also possible for students to perform a division that more directly demonstrates the equivalence of 0.999... and 1. Students are happy to believe that any fraction in which the denominator and numerator are equal is in fact 1, and that any fraction (including those equal to 1) can be converted to a decimal by dividing the numerator by the denominator as we did with 1/3 to get 0.333... Expressing numbers such as 4 as 4.000... is also likely to be accepted. A slight modification of the usual division algorithm can then be used to show that, for example, 4/4 (which of course is equal to 1), is in fact equal to 0.999...

$$\begin{array}{r} 0.9\ 9\ 9\ 9\ 9\ 9\ 9\ 9... \\ 4\overline{)4.0^{4}0^{4}0^{4}0^{4}0^{4}0^{4}0^{4}0...} \end{array}$$

The key difference between this and the usual procedure is that rather than saying, '4 into 4 goes 1 time' and placing 1 above the line in the ones place, we consider 4 into '40' as is usual if the first digit is less than the divisor. Then, rather than saying that is 10 we say that it is '9 with 4 remainder'.

This is all mathematically legitimate, but requires a degree of ease with notions of place value and with the division algorithm that stretches many students and is telling in terms of their number sense. Number sense has been described as a person's general understanding of number and operations, along with the ability and inclination to use this understanding in flexible ways to make mathematical judgements and to develop useful strategies for handling numbers and operations (McIntosh, Reys & Reys, 1993). Being able to employ familiar ideas in novel contexts and in non-routine ways, as in the example above, is of course indicative of number sense and conversely difficulty in such contexts suggests that the students' number sense may be lacking.

The link between the development of number sense and teaching for conceptual understanding is well established. Howden (1989) argues that because number sense builds on students' intuitive understandings of numbers, students with sound number sense come to believe that mathematics does indeed make sense; they are able to evaluate the reasonableness of their answers, and they develop confidence in their own mathematical ability. It could be argued that the students who do not believe the results of algebraic manipulations like that presented at the beginning of this paper, even though they can both follow and perform such procedures, are lacking in something very much akin to number sense — algebra sense perhaps. Howden (1989) goes on to cite research claiming that children with strong number sense have more positive self concepts with respect to mathematics and are more likely to choose to study mathematics in the future.

The 'proof' at the start of this paper relies on a technique that can be used to convert any recurring decimal to a fraction. For example:

$$\begin{aligned} 1\times 0.666\ldots &= 0.666\ldots \\ \therefore\ 10\times 0.666\ldots &= 6.666\ldots \\ \therefore\ 9\times 0.666\ldots &= 6 \\ \therefore\ 0.666\ldots &= \frac{6}{9} \end{aligned}$$

This relates to yet another way of explaining the equivalence of 0.999... and 1. Consider the following sequence of recurring decimals and their simplest fraction equivalents. As with 1/3, students can verify these equivalences by division:

$$\frac{1}{9} = 0.111\ldots$$

$$\frac{2}{9} = 0.222\ldots$$

$$\frac{3}{9} = 0.333\ldots$$

$\ldots$

$$\frac{7}{9} = 0.777\ldots$$

$$\frac{8}{9} = 0.888\ldots$$

$$\frac{9}{9} = 0.999\ldots$$

Of course $\frac{9}{9} = 1$, so $0.999\ldots = 1$

As an aside, for decimals with two repeating digits immediately after the decimal point, the second line involves multiplying by 100. For example:

$$\begin{aligned} 1\times 0.454545\ldots &= 0.454545\ldots \\ \therefore\ 100\times 0.454545\ldots &= 45.454545\ldots \\ \therefore\ 99\times 0.454545\ldots &= 45 \\ \therefore\ 0.454545\ldots &= \frac{45}{99} \end{aligned}$$

In general, the conversion of a repeating decimal where the repetition begins immediately after the decimal point, involves multiplying by $10n$ where n is the number of repeating digits. The general form of such repeating decimals expressed as fractions is thus:

$$\frac{\text{The repeating digits}}{10^n - 1}$$

A range of further explanations gets closer to the notion of infinity that is at the heart of the difficulty of the problem. Perhaps the simplest of these is the argument that one and only one of the following must be true:

0.999... < 1
0.999... = 1
0.999... > 1

The third is readily dismissed whereas the first implies that there exists a number between

0.999… and 1, because 0.999… being less than 1 implies that there must be a difference between them. Challenging students to name such a number is usually sufficient to resign them to the second alternative, but again there is the problem of students not really believing the result even though they follow the argument. Some find the impossibility of the first statement more readily acceptable by considering the subtraction:

$$\begin{array}{r} \scriptstyle 0\ 9\,9\,9\,9\,9\,9 \\ 1.\cancel{0}\cancel{0}\cancel{0}\cancel{0}\cancel{0}\cancel{0}\ldots \\ -\ 0.999999\ldots \\ \hline \end{array}$$

Starting at the right, as is usual for the standard algorithm, we need to decompose/rename from successive places until ultimately the question becomes 0.999… – 0.999… Of course, if there were an end, there would be a 1 to write down there. This fact can cause some debate!

An explanation that may appeal to students whose algebraic knowledge is relatively advanced involves the use of the sum of the terms in an infinite geometric progression.

$$0.9999\ldots = 0.9 + 0.09 + 0.009 + \ldots$$

The relevant formula for the sum to infinity of a geometric progression, a, ar, ar^2… is:

$$S_\infty = \frac{a}{(1-r)}$$

where a is the first term (in this case 0.9) and r is the common ratio (in this case 0.1)

$$\therefore S_\infty = \frac{0.9}{(1-0.1)}$$

$$\therefore S_\infty = 1$$

This is fine for those students inclined to believe formulae without question (or old enough to have studied sequences and series), but the problem of the extent to which students who can apply the formula actually believe the result remains. For those wanting further explanation (and the more in this category the better), the formula is derived by making use of the concept of limits and essentially means that the sum 0.9 + 0.09 + 0.009 + … (which is 0.999…) becomes arbitrarily close to 1. Another, perhaps more intuitive way to consider this is to consider the differences between the sum of the sequence and 1 if we stop at various points. For example, after one term the sum is 0.9, after two terms it is 0.99, and so on. The corresponding differences between these partial sums and 1 are 0.1, 0.01, 0.001… . Expressed as fractions they are

$$\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}\ldots$$

In this form students tend to be happy to believe that the difference becomes arbitrarily small and is, in the limit, 0.

Questions like that in the title of this paper not only deserve a serious and thoughtful answer but also can provide rich opportunities to explore important mathematics in a context in which the students are engaged and to make connections between aspects of mathematics that students may see as disparate. The fact that students ask such questions is encouraging as these students are trying to make sense of the mathematics they are dealing with. It is worth designing contexts in which they are likely to arise. In relation to 0.999… and 1, a simple way is to include both of these numbers in an activity requiring students to order a set of numbers. Controversy is virtually guaranteed.

As teachers we have a responsibility to have a range of ways in which we can assist our students to approach such problems and to arrive at resolutions that are satisfying to them as well as mathematically sound. I believe that the problem considered in this paper demonstrates that quite profound and inherently fascinating mathematics is accessible to students who have a sound number sense and deep conceptual understanding of very basic mathematics. This is one of many reasons why we should teach mathematics in ways that promote these attributes in students.

References

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