Goro Kato

The Heart of Cohomology

Springer

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by

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Marc groet 's morgens de dingen

Dag ventje met de fiets op de vaas met de bloem
ploem ploem
dag stoel naast de tafel
dag brood op de tafel
dag visserke-vis met de pijp
en
dag visserke-vis met de pet
pet en pijp
van het visserke-vis
goeiendag

Daa-ag vis
dag lieve vis
dag klein visselijn mijn

Paul van Ostaijen

Contents

Preface

The methods of (Co-) Homological Algebra provide a framework for Algebraic Geometry and Algebraic Analysis. The following two books were published during the late 1950's:

[CE] Cartan, H., Eilenberg, S., *Homological Algebra*, Princeton University Press (1956), and

[G] Godement, R., *Topologie Alg´ebraique et Th´eorie des Faisceaux*, Hermann, Paris (1958).

If you are capable of learning from either of these two books, I am afraid that *The Heart of Cohomology*, referred to hereafter as [THOC], is not for you. One of the goals of [THOC] is to provide young readers with elemental aspects of the algebraic treatment of cohomologies.

During the 1990's

[GM] Gelfand, S.I., Manin, Yu., I., *Methods of Homological Algebra*, Springer–Verlag, (1996), and

[W] Weibel, C.A., *An Introduction to Homological Algebra*, Cambridge University Press, (1994)

were published. The notion of a derived category is also treated in [GM] and [W].

In June, 2004, the author was given an opportunity to give a short course titled "Introduction to Derived Category" at the University of Antwerp, Antwerp, Belgium. This series of lectures was supported by the European Science Foundation, Scientific Programme of ESF. The handwritten lecture notes were distributed to attending members. [THOC] may be regarded as an expanded version of the Antwerp Lecture Notes. The style of [THOC] is more lecture-like and conversational. Prof. Fred van Oystaeyen is responsible for the title "The

Heart of Cohomology". In an effort to satisfy the intent of the title of this book, a more informal format has been chosen.

After each Chapter was written, the handwritten manuscript was sent to Dr. Daniel Larsson in Lund, Sweden, to be typed. As each Chapter was typed, we discussed his suggestions and questions. Dr. Larsson's contribution to [THOC] is highly appreciated.

We will give a brief introduction to each Chapter. In Chapter I we cover some of the basic notions in Category Theory. As general references we recommend

[BM] Mitchell, B., *The Theory of Categories*, Academic Press, 1965, and

[SH] Schubert, H., *Categories*, Springer-Verlag, 1972.

The original paper on the notion of a category

[EM] Eilenberg, S., MacLane, S., *General Theory of Natural Equivalences*, Trans. Amer. Math. Soc. **58**, (1945), 231–294

is still a very good reference. Our emphasis is on Yoneda's Lemma and the Yoneda Embedding. For example, for contravariant functors F and G from a category $\mathscr{C}$ to the category Set of sets, the Yoneda embedding

$$\tilde{} : \mathscr{C} \rightsquigarrow \hat{\mathscr{C}} := \mathsf{Set}^{\mathscr{C}^\circ}$$

gives an interpretation for the convenient notation $F(G)$ as

$$\tilde{F}(G) = \mathrm{Hom}_{\hat{\mathscr{C}}}(G, F)$$

(See Remark 5.)

We did not develop a cohomology theory based on the notion of a site. However, for a covering $\{U_i \to U\}$ of an object U in a site $\mathscr{C}$, the higher Čech cohomology with coefficient $F \in \mathrm{Ob}(\mathsf{Ab}^{\mathscr{C}^\circ})$ is the derived functor of the kernel of

$$\prod F(U_i) \xrightarrow{\mathrm{d}^0} \prod F(U_i \times U_j).$$

This higher Čech cohomology associated with the covering of U is the cohomology of the Čech complex

$$C^j(\{U_i \to U\}, F) = \prod F(U_{i_0} \times_U \cdots \times_U U_{i_j}).$$

One can continue the corresponding argument as shown in 3.4.3.

In Chapter II, the orthodox treatment of the notion of a derived functor for a left exact functor is given. In 2.11 through Note 15, a more general invariant than the cohomology is introduced. Namely for a sequence of objects and morphisms in an abelian category, when the composition $\mathrm{d}^2 = 0$ need not hold, we define two complexifying functors on the sequence. The cohomology of the complexified sequence is the notion of a precohomology generalizing cohomology. The half-exactness and the self-duality of precohomologies are proved. As a general reference for this Chapter,

[HS] Hilton, P.J., Stammbach, U., *A Course in Homological Algebra*, Graduate Texts in Mathematics, Springer-Verlag, 1971

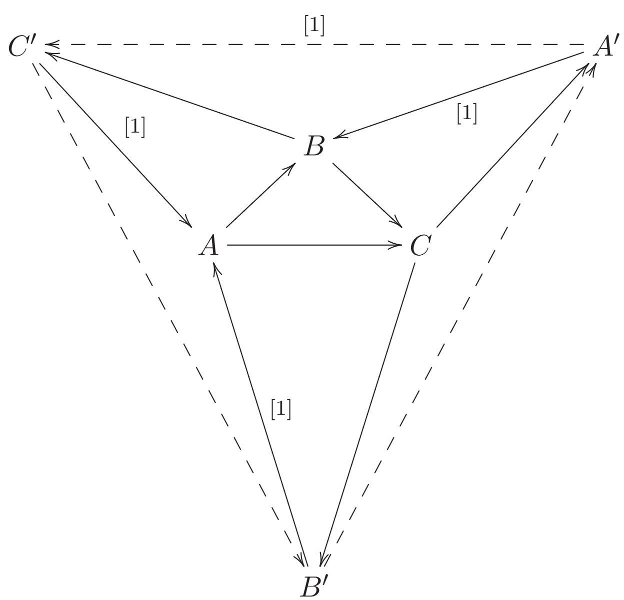
is also recommended.

In Chapter III, we focus on the spectral sequences associated with a double complex, the spectral sequences of composite functors, and the spectral sequences of hypercohomologies. For the *theory* of spectral sequences, in

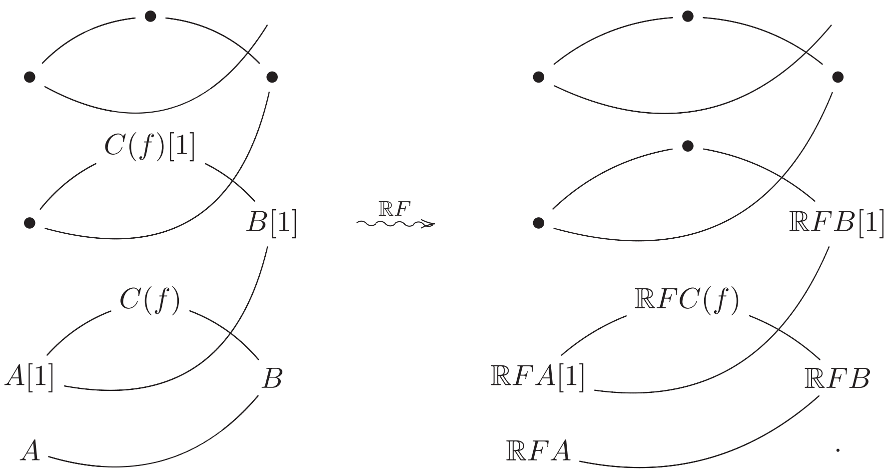
[LuCo] Lubkin, S., *Cohomology of Completions*, North-Holland, North-Holland Mathematics Studies 42, 1980

one can find the most general statements on abutments of spectral sequences. In [THOC], the interplay of the above three kinds of spectral sequences and their applications to sheaf cohomologies are given.

In Chapter IV, an elementary introduction to a derived category is given. Note that diagram (3.14) in Chapter IV comes from [GM]. The usual octahedral axiom for a triangulated category is replaced by the simpler (and maybe more natural) triangular axiom:



A schematic picture for the derived functors $\mathbb{R}F$ between derived categories carrying a distinguished triangle to distinguished triangle may be expressed as



As references for Chapter IV,

[HartRes] Hartshorne, R., *Residues and Duality*, Lecture Notes Math. 20, Springer-Verlag, 1966, and

[V] Verdier, J.L., *Catégories triangulées*, in *Cohomologie Étale*, SGA4$\frac{1}{2}$, Lecture Notes Math. 569, Springer-Verlag, 1977, 262–312.

need to be mentioned.

In Chapter V, applications of the materials in Chapters III and IV are given. The first half of Chapter V is focused on the background for the explicit computation of zeta invariances associated with the Weierstrass family. We wish to compute the homologies with compact supports of the closed fibre of the hyperplane

$$ZY^2 = 4X^3 - g_2XZ^2 - g_3Z^3$$

in $\mathbb{P}^2(\underline{A})$, $\underline{A} := \hat{\mathbb{Z}}_p[g_2, g_3]$, where X, Y, Z are homogeneous coordinates (or the open subfamily, i.e., the pre-image of $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3]_\Delta)$, i.e., localized at the discriminant $\Delta := g_2^3 - 27g_3^2$, $p \neq 2, 3$). Let U be the affine open family in the above fibre, i.e., "$Z = 1$". Then we are interested in a set of generators and relations for the $A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}$-module $\mathrm{H}^1_c(U, A^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$. For $\mathfrak{p}$ in the base $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3])$ (or $\mathrm{Spec}((\mathbb{Z}/p\mathbb{Z})[g_2, g_3]_\Delta)$, the universal spectral sequence is induced so as to compute the zeta function of the fibre over $\mathfrak{p}$ (or elliptic curve over $\mathfrak{p}$).

We also decided to include a letter from Prof. Dwork in 5.2.4 in Chapter V since we could not find the contents of this letter elsewhere.

In the second half of Chapter V, only some of the cohomological aspects of $\mathscr{D}$-modules are mentioned. None of the microlocal aspects of $\mathscr{D}$-modules are treated in this book. One may consider the latter half materials of Chapter V as examples and exercises of the spectral sequences and derived categories in Chapters III and IV.

Lastly, I would like to express my gratitude to my mathematician friends in the U.S.A., Japan and Europe. I will not try to list the names of these people here fearing that the names of significant people might be omitted. However, I would like to mention the name of my teacher and Ph.D. advisor, Prof. Saul Lubkin. I would like to apologize to him, however, because I was not able to learn as much as he exposed me to during my student years in the late 1970's. (I wonder where my Mephistopheles is.) In a sense, this book is my humble delayed report to Prof. Lubkin.

Tomo enpouyori kitari
mata tanoshi karazuya. . .

Goro Kato

Thanksgiving Holiday with my Family and Friends, 2005

Chapter 1

CATEGORY

1.1 Categories and Functors

The notion of a category is a concise concept shared among "groups and group homomorphisms", "set and set-theoretic mappings", "topological spaces and continuous mappings", e t c.

Definition 1. A *category* $\mathscr{C}$ consists of *objects*, denoted as $X, Y, Z, \ldots$, and *morphisms*, denoted as $f, g, \phi, \psi, \alpha, \beta, \ldots$. For objects X and Y in the category $\mathscr{C}$, there is induced the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms from X to Y. If $\phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ we write $\phi : X \to Y$ or $X \xrightarrow{\phi} Y$. Then, for $\phi : X \to Y$ and $\psi : Y \to Z$, the composition $\psi \circ \phi : X \to Z$ is defined. Furthermore, for $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \xrightarrow{\gamma} W$, the associative law $\gamma \circ (\psi \circ \phi) = (\gamma \circ \psi) \circ \phi$ holds. For each object X there exists a morphism $1_X : X \to X$ such that for $f : X \to Y$ and for $g : Z \to X$ we have $f \circ 1_X = f$ and $1_X \circ g = g$. Lastly, the sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ are pairwise disjoint. Namely, if $\mathrm{Hom}_{\mathscr{C}}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(X', Y')$, then $X = X'$ and $Y = Y'$.

Note 1. When X is an object of a category $\mathscr{C}$ we also write $X \in \mathrm{Ob}(\mathscr{C})$, the class of objects in $\mathscr{C}$. Note that a category is said to be *small* if $\mathrm{Ob}(\mathscr{C})$ is a set.

Example 1. The category Ab of abelian groups consists of abelian groups and group homomorphisms as morphisms. The category Set of sets consists of sets and set-theoretic maps as morphisms. Next let T be a topological space. Then there is an induced category $\mathscr{T}$ consisting of the open sets of T as objects. For open sets $U, V \subset T$, the induced set $\mathrm{Hom}_{\mathscr{T}}(U, V)$ of morphisms from U and V consists of the inclusion map $\iota : U \hookrightarrow V$ if $U \subset V$, and $\mathrm{Hom}_{\mathscr{T}}(U, V)$ an empty set if $U \not\subset V$.

Remark 1. For the category Ab we have the familiar element-wise definitions of the kernel and the image of a group homomorphism f from a group G to a group H. We also have the notions of a monomorphism, called an injective homomorphism, and of an epimorphism, called a surjective homomorphism in the category Ab. For a general category $\mathscr{C}$ we need to give appropriate definitions without using elements for the above mentioned concepts. For example, $\phi : X \to Y$ in $\mathscr{C}$ is said to be an *epimorphism* if $f \circ \phi = g \circ \phi$ implies $f = g$ where $f, g : Y \to Z$. (This definition of an epimorphism is reasonable since the agreement $f \circ \phi = g \circ \phi$ only on the set-theoretic image of ϕ guarantees that $f = g$.) Similarly, $\phi : X \to Y$ is said to be a *monomorphism* if $\phi \circ f = \phi \circ g$ implies $f = g$ where $f, g : W \to X$. (This is reasonable since there can not be two different paths from W to Y.) In order to give a categorical definition of an image of a morphism, we need to define the notion of a subobject. Let $W \xrightarrow{\phi} X$ and $W' \xrightarrow{\phi'} X$ be monomorphisms. Then define a pre-order $(W', \phi') \leq (W, \phi)$ if and only if there exists a morphism $\psi : W' \to W$ satisfying $\phi \circ \psi = \phi'$. Notice that ψ is a uniquely determined monomorphism. If $(W, \phi) \leq (W', \phi')$ also holds, we have a monomorphism $\psi' : W \to W'$ satisfying $\phi' \circ \psi' = \phi$ and so $\phi \circ \psi \circ \psi' = \phi' \circ \psi' = \phi = \phi \circ 1_W$. Since ϕ is a monomorphism we have $\psi \circ \psi' = 1_W$. Similarly, we also have $\psi' \circ \psi = 1_{W'}$. This means that ψ is an isomorphism, and (W, ϕ), (W', ϕ') are said to be equivalent. A *subobject* of X is defined as an equivalence class of such pairs (W, ϕ). A categorical, i.e., element-free, definition of the image of a morphism $\phi : X \to Y$ may be given as follows. Consider a factorization of ϕ

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ & {\scriptstyle \phi'}\searrow & \uparrow {\scriptstyle \iota} \\ & & Y' \end{array} \tag{1.1}$$

where (Y', ι) is a subobject of Y. For another such factorization (Y'', ι'), if there exists a morphism $j : Y' \to Y''$ satisfying $\iota = \iota' \circ j$, then (Y', ι) is said to be the *image* of ϕ. Intuitively speaking, shrink Y as much as possible to Y' so that factorization is still possible. Namely, the image of ϕ is the smallest subobject (Y', ι) to satisfy the commutative diagram (1.1). On the other hand, the *kernel* of $\phi : X \to Y$ can be characterized as the largest subobject (X', ι) of satisfying $\phi \circ \iota = 0$ in

$$\begin{array}{ccc} X & \xrightarrow{\phi} & Y \\ {\scriptstyle \iota}\uparrow & \nearrow {\scriptstyle \phi'} & \\ X' & & \end{array} \tag{1.2}$$

1.1.1 Cohomology in Ab

For a sequence

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$$

in Ab, the *cohomology group* at Y is defined as the quotient group of Y

$$\ker\psi \big/ \operatorname{im}\phi \tag{1.3}$$

provided $\operatorname{im}\phi \subset \ker\psi$, i.e., for $y = \phi(x) \in \operatorname{im}\phi$ we have $\psi(y) = 0$, or in still other words, $\psi(y) = \psi(\phi(x)) = (\psi \circ \phi)(x) = 0$.

1.1.2 The functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$

Let us take a close look at the set of morphisms $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ in Definition 1. First consider $\mathrm{Hom}_{\mathscr{C}}(X, X)$. Recall that there is a special morphism from X to X, call it 1_X, satisfying the following. For any $\phi : X \to Y$ and $\psi : Z \to X$ we have $1_X \circ \psi = \psi$ and $\phi \circ 1_X = \phi$ in

$$Z \xrightarrow{\psi} X \xrightarrow{1_X} X \xrightarrow{\phi} Y. \tag{1.4}$$

Then 1_X is said to be an *identity morphism* as in Definition 1, (i).

Next delete Y in the expression $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ to get $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$. Then, regard $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$ as an assignment

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} &\longrightarrow \mathsf{Set} \\ Y &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{aligned} \tag{1.5}$$

Similarly we can consider

$$\begin{aligned} \mathrm{Hom}_{\mathscr{C}}(\cdot, Y) : \mathscr{C} &\longrightarrow \mathsf{Set} \\ X &\longmapsto \mathrm{Hom}_{\mathscr{C}}(X, Y). \end{aligned} \tag{1.6}$$

That is, when you substitute Y in the deleted spot of $\mathrm{Hom}_{\mathscr{C}}(X, \cdot)$, you get the set $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ of morphisms. For two objects Y and Y' we have two sets $\mathrm{Hom}_{\mathscr{C}}(X, Y)$ and $\mathrm{Hom}_{\mathscr{C}}(X, Y')$. Then for a morphism $\beta : Y \to Y'$ consider the diagram

$$\begin{array}{ccccc} & & X & & \\ & \stackrel{\phi}{\swarrow} & & \stackrel{\beta\circ\phi}{\searrow} & \\ Y & & \xrightarrow{\quad\beta\quad} & & Y' \end{array} \tag{1.7}$$

This diagram indicates that for $\phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$, we get $\beta \circ \phi \in \mathrm{Hom}_{\mathscr{C}}(X, Y')$. Schematically, we express this situation as:

$$\begin{array}{rcll} \beta : Y & \longrightarrow & Y' & \text{in } \mathscr{C} \\ & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{C}}(X,\cdot)} & & \\ \mathrm{Hom}_{\mathscr{C}}(X, \beta) : \mathrm{Hom}_{\mathscr{C}}(X, Y) & \longrightarrow & \mathrm{Hom}_{\mathscr{C}}(X, Y') & \text{in Set} \end{array} \tag{1.8}$$

where $\mathrm{Hom}_{\mathscr{C}}(X, \beta)(\phi) := \beta \circ \phi$.

On the other hand, when X is deleted from $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, we get (1.6). But for $X \xrightarrow{\alpha} X'$, i.e., considering

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & {\scriptstyle \psi\circ\alpha} \searrow \quad \swarrow {\scriptstyle \psi} & \\ & Y & \end{array} \tag{1.9}$$

$\psi \in \mathrm{Hom}_{\mathscr{C}}(X', Y)$ induces $\psi \circ \alpha \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$. Schematically,

$$\begin{array}{rcll} \alpha : X & \longrightarrow & X' & \text{in } \mathscr{C} \\ & \big\downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{C}}(\cdot,Y)} & & \\ \mathrm{Hom}_{\mathscr{C}}(\alpha, Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}}(X', Y) & \text{in Set} \end{array} \tag{1.10}$$

Notice that the direction of the morphism in (1.10) is changed as compared with $\mathrm{Hom}_{\mathscr{C}}(X, \beta)$ in (1.8).

Definition 2. Let $\mathscr{C}$ and $\mathscr{C}'$ be categories. A *covariant functor* from $\mathscr{C}$ to $\mathscr{C}'$ denoted as $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$, is an assignment of an object FX in $\mathscr{C}'$ to each object X in $\mathscr{C}$ and a morphism $F\alpha$ from FX to FX' to each morphism $\alpha : X \to X'$ in $\mathscr{C}$ satisfying:

(Func1) For $X \xrightarrow{\alpha} X' \xrightarrow{\alpha'} X''$ in $\mathscr{C}$ we have

$$F(\alpha' \circ \alpha) = F\alpha' \circ F\alpha.$$

(Func2) For $1_X : X \to X$ we have $F1_X = 1_{FX} : FX \to FX$.

Condition (Func1) may schematically be expressed as the commutativity of

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & X' \\ & {\scriptstyle \alpha'\circ\alpha} \searrow & \downarrow {\scriptstyle \alpha'} \\ & & X'' \end{array} \qquad\qquad \begin{array}{ccc} FX & \xrightarrow{F\alpha} & FX' \\ & {\scriptstyle F(\alpha'\circ\alpha)} \searrow & \downarrow {\scriptstyle F\alpha'} \\ & & FX'' \end{array} \tag{1.11}$$

in $\mathscr{C}$ in $\mathscr{C}'$

Example 2. In Definition 2, let $\mathscr{C}' = \mathsf{Set}$ and let $F = \mathrm{Hom}_{\mathscr{C}}(X, \cdot)$. Then one notices from (1.8) that $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$ is a covariant functor.

Note 2. Similarly, a *contravariant functor* $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ can be defined as in Definition 2 with the following exception: For $\alpha : X \to X'$ in $\mathscr{C}$, $F\alpha$ is a morphism from FX' to FX in $\mathscr{C}'$, i.e., as in (1.10) the direction of the morphism is changed. Notice that $\mathrm{Hom}_{\mathscr{C}}(\cdot, Y)$ is a contravariant functor from $\mathscr{C}$ to Set.

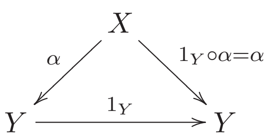
Before we begin the next topic, let us confirm that the covariant functor $\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$ satisfies Condition (Func2) of Definition 2. To demonstrate this: for $1_Y : Y \to Y$, indeed

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

is to be the identity morphism on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$, i.e.,

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) = 1_{\mathrm{Hom}_{\mathscr{C}}(X,Y)}.$$

Let $\alpha \in \mathrm{Hom}_{\mathscr{C}}(X, Y)$ be an arbitrary morphism. Then consider



(1.12)

which is a special case of (1.7). As shown in (1.8), the definition of

$$\mathrm{Hom}_{\mathscr{C}}(X, 1_Y) : \mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{C}}(X, Y)$$

is $\alpha \mapsto 1_Y \circ \alpha = \alpha$. Namely, $\mathrm{Hom}_{\mathscr{C}}(X, 1_Y)$ is an identity on $\mathrm{Hom}_{\mathscr{C}}(X, Y)$.

1.2 Opposite Category

Next, we will define the notion of an *opposite category* (or *dual category*). Let $\mathscr{C}$ be a category. Then the opposite category $\mathscr{C}^\circ$ has the same objects as $\mathscr{C}$. This means that the dual object X° in $\mathscr{C}^\circ$ of an object X in $\mathscr{C}$ satisfies $X^\circ = X$. We will use the same X even when X is an object of $\mathscr{C}^\circ$. Let X and Y be objects in $\mathscr{C}^\circ$, then the set of morphisms from X to Y in $\mathscr{C}^\circ$ is defined as the set of morphisms from Y to X in $\mathscr{C}$, i.e.,

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X, Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X). \tag{2.1}$$

Note that $\mathscr{C}^\circ$ is also called the *dual category* of $\mathscr{C}$. Recall that

$$\mathrm{Hom}_{\mathscr{C}}(X, \cdot) : \mathscr{C} \rightsquigarrow \mathsf{Set}$$

is a covariant functor. Let us replace $\mathscr{C}$ by $\mathscr{C}^\circ$. Then we have

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot) : \mathscr{C}^\circ \rightsquigarrow \mathsf{Set}.$$

Let $Y \xrightarrow{\phi} Y'$ be a morphism in $\mathscr{C}$. Then in $\mathscr{C}^\circ$ we have $Y \xleftarrow{\phi^\circ} Y'$. The covariant functor $\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot)$ takes $Y \xleftarrow{\phi^\circ} Y'$ in $\mathscr{C}^\circ$ without changing the direction of ϕ° to

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,Y) \longleftarrow \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y')$$

in Set. From (2.1) we get

$$\mathrm{Hom}_{\mathscr{C}^\circ}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(Y,X) \longleftarrow \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y') = \mathrm{Hom}_{\mathscr{C}}(Y',X)\ .$$

Schematically, we have

$$\begin{array}{lccc} \text{In } \mathscr{C}^\circ : & Y & \xleftarrow{\ \phi^\circ\ } & Y'\ . \\ & {\scriptstyle\circ}\big\downarrow & & \big\uparrow{\scriptstyle\circ} \\ \text{In } \mathscr{C} : & Y & \xrightarrow{\ \phi\ } & Y' \end{array} \tag{2.2a}$$

Applying $\mathrm{Hom}_{\mathscr{C}^\circ}(X,\cdot)$ to the top row and $\mathrm{Hom}_{\mathscr{C}}(\cdot,X)$ to the bottom row, we get:

$$\begin{array}{ccc} \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}^\circ}(X,Y') \\ \Big\| & & \Big\| \\ \mathrm{Hom}_{\mathscr{C}}(Y,X) & \longleftarrow & \mathrm{Hom}_{\mathscr{C}}(Y',X) \end{array} \tag{2.2b}$$

in Set. Generally, for a covariant functor $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$, there is induced a contravariant functor $F : \mathscr{C}^\circ \rightsquigarrow \mathscr{C}'$. On the other hand, $F : \mathscr{C} \rightsquigarrow \mathscr{C}'^\circ$ becomes contravariant.

1.2.1 Presheaf on $\mathscr{T}$

In Example 1, we defined the category $\mathscr{T}$ associated with a topological space T. Let us consider a contravariant functor F from $\mathscr{T}$ to a category $\mathscr{A}$. Namely, for $U \hookrightarrow V$ in $\mathscr{T}$, we have $FU \leftarrow FV$ in $\mathscr{A}$. (As noted, $F : \mathscr{T}^\circ \rightsquigarrow \mathscr{A}$ is a covariant functor.) Then F is said to be a *presheaf* defined on $\mathscr{T}$ with values in $\mathscr{A}$. In the category of presheaves on $\mathscr{T}$

$$\hat{\mathscr{T}} := \mathscr{A}^{\mathscr{T}^\circ}, \tag{2.3}$$

an object is a covariant functor (presheaf) from $\mathscr{T}^\circ$ to $\mathscr{A}$, and a morphism f of presheaves F and G is defined as follows. To every object U of $\mathscr{T}$, f assigns a morphism

$$f_U : FU \to GU \tag{2.4}$$

in $\mathscr{A}$. Generally, for categories $\mathscr{C}$ and $\mathscr{C}'$, let

$$\hat{\mathscr{C}} = \mathscr{C}'^{\mathscr{C}} \tag{2.5}$$

be the category of (covariant) functors as its objects. For functors F and G, a morphism $f : F \to G$ is called a *natural transformation* from F to G and is defined as an assignment $f_U : FU \to GU$ for an object U in $\mathscr{C}$. Additionally f must satisfy the following condition: for every $U \xrightarrow{\alpha} V$ in $\mathscr{C}$, the diagram

$$\begin{array}{ccc} FU & \xrightarrow{f_U} & GU \\ \downarrow{\scriptstyle F\alpha} & & \downarrow{\scriptstyle G\alpha} \\ FV & \xrightarrow{f_V} & GV \end{array} \tag{2.6a}$$

commutes, i.e., $f_V \circ F\alpha = G\alpha \circ f_U$ in $\mathscr{C}'$. Therefore, a morphism $f : F \to G$ in $\hat{\mathscr{T}} = \mathscr{A}^{\mathscr{T}^\circ}$ must satisfy the following in addition to (2.4). For $\iota : U \hookrightarrow V$ in $\mathscr{T}$ (i.e., $U \hookleftarrow V$ in $\mathscr{T}^\circ$),

$$\begin{array}{ccc} FU & \xrightarrow{f_U} & GU \\ {\scriptstyle F\iota}\uparrow & & \uparrow{\scriptstyle G\iota} \\ FV & \xrightarrow{f_V} & GV \end{array} \tag{2.6b}$$

must commute. Important examples of $\hat{\mathscr{T}}$ are the cases when $\mathscr{A} = \mathsf{Set}$ and $\mathscr{A} = \mathsf{Ab}$. We will return to this topic when the notion of a site is introduced.

1.3 Forgetful Functors

Let A be an abelian group. By forgetting the abelian group structure, A can be regarded as just a set. Namely, we have an assignment $S : \mathsf{Ab} \rightsquigarrow \mathsf{Set}$. For a group homomorphism $\phi : A \to B$ in Ab, assign the set-theoretic map $S\phi : SA \to SB$. One may wish to check axioms (Func1) and (Func2) of Definition 2 for the assignment S. Consequently S is a covariant functor from Ab to Set. This functor S is said to be a forgetful functor from Ab to Set.

Definition 3. Let $\mathscr{C}'$ and $\mathscr{C}$ be categories. Then $\mathscr{C}'$ is a *subcategory* of $\mathscr{C}$ when the following conditions are satisfied.

(Subcat1) $\mathrm{Ob}(\mathscr{C}') \subset \mathrm{Ob}(\mathscr{C})$ and for all objects X and Y in $\mathscr{C}'$,

$$\mathrm{Hom}_{\mathscr{C}'}(X, Y) \subset \mathrm{Hom}_{\mathscr{C}}(X, Y).$$

(Subcat2) The composition of morphisms in $\mathscr{C}'$ is coming from the composition of morphisms in $\mathscr{C}$, and for all objects X in $\mathscr{C}'$ the identity morphisms 1_X in $\mathscr{C}'$ are also identity morphisms in $\mathscr{C}$.

Example 3. Let $\mathcal{V}'$ be the category of finite-dimensional vector spaces over a field ℓ and let $\mathcal{V}$ be the category of vector spaces over ℓ and where the morphisms are the ℓ-linear transformations. Then $\mathcal{V}'$ is a subcategory of $\mathcal{V}$. Let Top be the category of topological spaces where the morphisms are continuous mappings. Then Top is a subcategory of Set.

Remark 2. Note that we have $\mathrm{Hom}_{\mathcal{V}'}(X,Y) = \mathrm{Hom}_{\mathcal{V}}(X,Y)$, since the ℓ-linearity has nothing to do with dimensions. In general, when a subcategory $\mathscr{C}'$ of a category $\mathscr{C}$ satisfies $\mathrm{Hom}_{\mathscr{C}'}(X,Y) = \mathrm{Hom}_{\mathscr{C}}(X,Y)$ for all X and Y in $\mathscr{C}'$, $\mathscr{C}'$ is said to be a *full subcategory* of $\mathscr{C}$.

1.4 Embedddings

Let $\mathscr{B}$ and $\mathscr{C}$ be categories. Even though $\mathscr{B}$ is not a subcategory of $\mathscr{C}$, one can ask whether $\mathscr{B}$ can be embedded in $\mathscr{C}$ (whose definition will be given in the following). Let F be a covariant functor from $\mathscr{B}$ to $\mathscr{C}$. Then for $f : X \to Y$ in $\mathscr{B}$ we have $FX \to FY$ in $\mathscr{C}$. Namely, for an element f of $\mathrm{Hom}_{\mathscr{B}}(X,Y)$ we obtain Ff in $\mathrm{Hom}_{\mathscr{C}}(FX,FY)$. That is we have the following map $\bar{F}$:

$$\begin{aligned} \bar{F} : \mathrm{Hom}_{\mathscr{B}}(X,Y) &\longrightarrow \mathrm{Hom}_{\mathscr{C}}(FX,FY) \\ f &\longmapsto \bar{F}(f) = Ff \end{aligned} \tag{4.1}$$

If $\bar{F}$ is injective, $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to be *faithful*, and if $\bar{F}$ is surjective, F is said to be *full*. Furthermore, F is said to be an *embedding* (or *imbedding*) if $\bar{F}$ is not only injective on morphisms, but also F is injective on objects. That is, $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to be an embedding if F is a faithful functor and if $FX = FY$ implies $X = Y$. Then $\mathscr{B}$ may be regarded as a subcategory of $\mathscr{C}$. We also say that $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is *fully faithful* when F is full and faithful. A functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is said to *represent* $\mathscr{C}$ when the following condition is satisfied: For every object X' of $\mathscr{C}$ there exists an object X in $\mathscr{B}$ so that there exists an isomorphism from FX to X'. If a fully faithful functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ represents $\mathscr{C}$ then F is said to be an *equivalence*. Furthermore, an equivalence F is said to be an *isomorphism* if F induces an injective correspondence between the objects of $\mathscr{B}$ and $\mathscr{C}$. The notion of an equivalence F can be characterized by the following.

Proposition 3. *A functor $F : \mathscr{B} \rightsquigarrow \mathscr{C}$ is an equivalence if and only if there exists a functor $F' : \mathscr{C} \rightsquigarrow \mathscr{B}$ satisfying*

(Eqv) *$F' \circ F$ and $F \circ F'$ are isomorphic to the identity functors $1_{\mathscr{B}}$ and $1_{\mathscr{C}}$, respectively.*

Proof. Let $f : Z \to Z'$ be a morphism in $\mathscr{C}$. Since F represents $\mathscr{C}$, there are objects X and X' in $\mathscr{B}$ so that $FX \xrightarrow{i} Z$ and $FX' \xrightarrow{j} Z'$ are isomorphisms in $\mathscr{C}$.

Then we have the morphism $j^{-1} \circ f \circ i : FX \to FX'$. Define $\tilde{f} := j^{-1} \circ f \circ i$. Since F is fully faithful there exists a unique morphism $\tilde{f}' : X \to X'$ in $\mathscr{B}$ satisfying $F\tilde{f}' = \tilde{f}$. Then define $F'f := \tilde{f}'$. Namely, we have $F'Z = X$ and $F'Z' = X'$. Note that F' becomes a functor from $\mathscr{C}$ to $\mathscr{B}$. From the commutative diagram

$$\begin{array}{ccc} FX & \xrightarrow[i]{\approx} & Z \\ {\scriptstyle \tilde{f}:=j^{-1}\circ f\circ i}\downarrow & & \downarrow{\scriptstyle f} \\ FX' & \xrightarrow[j]{\approx} & Z' \end{array} \tag{4.2}$$

in $\mathscr{C}$, we get the commutative diagram in $\mathscr{B}$

$$\begin{array}{ccc} F'FX & \xrightarrow[F'i]{\approx} & F'Z = X \\ \downarrow & & \downarrow{\scriptstyle F'f:=\tilde{f}'} \\ F'FX' & \xrightarrow[F'j]{\approx} & F'Z' = X'. \end{array} \tag{4.3}$$

From the definition of F', i.e., $F'Z = X$ and (4.2), we also get

$$\begin{array}{ccc} FF'Z & \xrightarrow[i]{\approx} & Z \\ \downarrow & & \downarrow{\scriptstyle f} \\ FF'Z' & \xrightarrow[j]{\approx} & Z'. \end{array} \tag{4.4}$$

We obtain $F' \circ F \approx 1_{\mathscr{B}}$ and $F \circ F' \approx 1_{\mathscr{C}}$.

Conversely, assume (Eqv). For an object Z of $\mathscr{C}$ we have an isomorphism $(F \circ F')Z \xrightarrow{\approx} 1_{\mathscr{C}}Z = Z$. Let $X = F'Z$. Then $FX \xrightarrow{\approx} Z$. Therefore, F represents $\mathscr{C}$. Consider $\bar{F}$ of (4.1), i.e.,

$$\bar{F} : \mathrm{Hom}_{\mathscr{B}}(X, X') \to \mathrm{Hom}_{\mathscr{C}}(FX, FX').$$

Suppose that $\bar{F}f = \bar{F}g$ for $f, g \in \mathrm{Hom}_{\mathscr{B}}(X, X')$. We have $Ff = Fg$ which implies $F'Ff = F'Fg$. Since $F' \circ F \xrightarrow{\approx} 1_{\mathscr{B}}$, $f = g$. Therefore F is faithful. Let $\phi \in \mathrm{Hom}_{\mathscr{C}}(FX, FX')$. Since F represents $\mathscr{C}$, we have isomorphisms $F(F'FX) \xrightarrow[i]{\approx} FX$ and $F(F'FX') \xrightarrow[j]{\approx} FX'$. That is, we have the commutative diagram

$$\begin{array}{ccc} FF'FX & \xrightarrow[i]{\approx} & FX \\ \downarrow{\scriptstyle F(F'\phi)} & & \downarrow{\scriptstyle \phi} \\ FF'FX' & \xrightarrow[j]{\approx} & FX'. \end{array} \tag{4.5}$$

Then $F'\phi : F'FX \to F'FX'$, i.e., $F'\phi \in \mathrm{Hom}_{\mathscr{B}}(X, X')$ satisfying

$$\bar{F}(F'\phi) = (F \circ F')\phi = 1_{\mathscr{C}}\phi = \phi.$$

Therefore, F is full.

Remark 3. When there is an equivalence $F : \mathscr{B} \rightsquigarrow \mathscr{C}$, $\mathscr{B}$ may be identified with $\mathscr{C}$ in the following sense. If there are objects X and X' in $\mathscr{B}$ having isomorphisms $FX \xrightarrow{i} Z$ and $FX' \xrightarrow{j} Z$ then we get the isomorphisms $F'FX \xrightarrow{F'i} F'Z$ and $F'FX' \xrightarrow{F'j} F'Z$. Namely,

$$X \xrightarrow[F'i]{\approx} F'Z \xleftarrow[F'j]{\approx} X' \;.$$

Considering Z' as isomorphic to Z we can conclude that there is a bijective correspondence between isomorphic classes of $\mathscr{B}$ and $\mathscr{C}$.

1.5 Representable Functors

First recall from (1.9) that $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a contravariant functor from $\mathscr{C}$ to Set. Let G also be a contravariant functor from $\mathscr{C}$ to Set. Namely, $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ and G are objects of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$ as in (2.5) and (2.6a). For $G \in \mathrm{Ob}(\hat{\mathscr{C}})$, if there exists an object X in $\mathscr{C}$ so that $\mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is isomorphic to G in the category $\hat{\mathscr{C}}$, then G is said to be a *representable functor*. We also say that G and $\widetilde{X} := \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ are *naturally equivalent*. That is, there is a natural transformation $\alpha : \widetilde{X} \to G$ (i.e., α is a morphism in $\hat{\mathscr{C}}$) which gives an isomorphism for every object Y in $\mathscr{C}$

$$\alpha_Y : \widetilde{X}(Y) = \mathrm{Hom}_{\mathscr{C}}(Y, X) \to GY. \tag{5.1}$$

Such an α is said to be a *natural equivalence*.

1.5.1 Yoneda's Lemma

Let F be an arbitrary contravariant functor from a category $\mathscr{C}$ to Set. For two objects F and $\widetilde{X} = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$, consider the set $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ of all morphisms in $\hat{\mathscr{C}}$ from $\widetilde{X}$ to F, i.e., $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ is the set of all the natural transformations from $\widetilde{X}$ to F. The Yoneda Lemma asserts that there is an isomorphism (i.e., a bijection) between the sets $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ and FX. If an element of $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ is written vertically as

$$\begin{array}{c} F \\ \uparrow \\ \widetilde{X} \end{array} \tag{5.2}$$

the reader with a scheme-theoretic background might consider such a morphism as (5.2) as an $\widetilde{X}$-rational point on F, suggesting $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F) \approx F(\widetilde{X})$. As the functor $\widetilde{\ }: \mathscr{C} \rightsquigarrow \hat{\mathscr{C}}$ will later be shown to be an embedding, the identification of $\widetilde{X}$ with X would be appropriate. Namely, FX might be interpreted as the set of all the X-rational points on F.

Proposition 4 (Yoneda's Lemma)**.** *For a contravariant functor F from a category $\mathscr{C}$ to the category* Set *of sets, there is a bijection*

$$\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F) \approx FX, \tag{5.3}$$

where X is an arbitrary object of $\mathscr{C}$.

Proof. Let $r \in \mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$, i.e., $r : \widetilde{X} \to F$ is a natural transformation. For X itself, we have

$$r_X : \widetilde{X}X \to FX. \tag{5.4}$$

Then for $1_X \in \widetilde{X}X = \mathrm{Hom}_{\mathscr{C}}(X, X)$, $r_X(1_X)$ is an element of FX. Namely, we obtain a map α from $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ to FX defined by $\alpha(r) = r_X(1_X)$. We will show that this map α is a bijection. Define a map from FX to $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$ as follows. Let $x \in FX$. Then we need a natural transformation ϕ_x from $\widetilde{X}$ to F. That is, for an arbitrary object Y of $\mathscr{C}$ we need a map $\phi_{x,Y}$ from $\widetilde{X}Y = \mathrm{Hom}_{\mathscr{C}}(Y, X)$ to FY. Consider the following commutative diagrams:

$$\begin{array}{ccc} Y & & \\ {\scriptstyle f}\downarrow & \searrow^{f = 1_X \circ f} & \\ X & \xrightarrow[1_X]{} & X \end{array} \tag{5.5a}$$

$$\begin{array}{ccc} \widetilde{X}X = \mathrm{Hom}_{\mathscr{C}}(X, X) & \longrightarrow & FX \\ \downarrow{\scriptstyle \mathrm{Hom}_{\mathscr{C}}(f, X)} & & \downarrow{\scriptstyle Ff} \\ \widetilde{X}Y = \mathrm{Hom}_{\mathscr{C}}(Y, X) & \longrightarrow & FY. \end{array} \tag{5.5b}$$

Then for $f \in \widetilde{X}Y = \mathrm{Hom}_{\mathscr{C}}(Y, X)$, $Ff : FX \to FY$ gives $(Ff)(x) \in FY$. That is, for $x \in FX$, the map $\phi_{x,Y}$ from $\widetilde{X}Y \to FY$ is given by $f \mapsto (Ff)(x)$. We are ready to compute the compositions of these maps. First we will prove $\alpha(\phi_x) = x$. By definition of α, $\alpha(\phi_x) = \phi_{x,X}(1_X)$. That is, for $\phi_x : \widetilde{X} \to F$, $\phi_{x,X}$ is the map from $\widetilde{X}X \to FX$. Then, by the definition of $\phi_{x,X}$, we have $\phi_{x,X}(1_X) = (F1_X)(x) = 1_{FX}(x) = x$. Conversely, let $r \in \mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$. Then $\alpha(r) = r_X(1_X) \in FX$. We need to show $\phi_{r_X(1_X)} = r$ as natural transformations in $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X}, F)$. That is, for an arbitrary object Y in $\mathscr{C}$, we

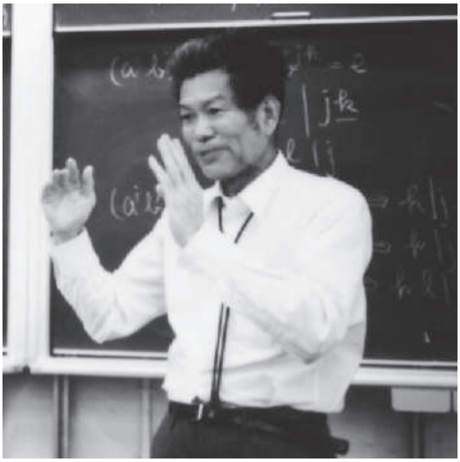


Figure 1.1. Nobuo Yoneda. Provided by Iwanami-Shoten, Inc.

must show $\phi_{r_X(1_X),Y} = r_Y$ as maps from $\widetilde{X}Y = \mathrm{Hom}_{\mathscr{C}}(Y,X)$ to FY. Now we will compute: for $f \in \widetilde{X}Y = \mathrm{Hom}_{\mathscr{C}}(Y,X)$, the definition of $\phi_{x,Y}$ implies $\phi_{r_X(1_X),Y}(f) = (Ff)(r_X(1_X))$. In (5.5b) we regard $(Ff)(r_X(1_X))$ as the clockwise image of $1_X \in \widetilde{X}X$. Next, we will consider the counterclockwise route of (5.5b) for $1_X \in \widetilde{X}X$. First (5.5a) implies that

$$\mathrm{Hom}_{\mathscr{C}}(f,X)(1_X) = f \in \widetilde{X}Y.$$

For the given $r \in \mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X},F)$ the commutativity of (5.5b) implies

$$r_Y(f) = (Ff)(r_X(1_X))$$

for any $Y \in \mathrm{Ob}(\mathscr{C})$ and for any $f \in \widetilde{X}Y$.

Note 5. Notice that the Yoneda Lemma is also valid for a covariant functor $F : \mathscr{C} \rightsquigarrow \mathsf{Set}$ and $\widetilde{X} = \mathrm{Hom}_{\mathscr{C}}(X,\cdot)$.

Remark 4. For the Yoneda bijection $\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X},F) \approx FX$, consider the case where the contravariant functor F is representable and represented by $X' \in \mathrm{Ob}(\mathscr{C})$. Namely, we have

$$\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X},F) \approx \mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X},\widetilde{X}') \approx \widetilde{X}'X \approx FX.$$

Since $\widetilde{X}'X = \mathrm{Hom}_{\mathscr{C}}(X,X')$,

$$\mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{X},\widetilde{X}') \approx \mathrm{Hom}_{\mathscr{C}}(X,X'). \tag{5.6}$$

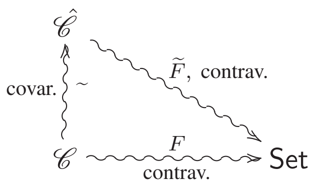
Notice that $\widetilde{X} = \mathrm{Hom}_{\mathscr{C}}(\cdot,X)$ is a contravariant functor from $\mathscr{C}$ to Set but the functor $\widetilde{\ }$ from $\mathscr{C}$ to $\hat{\mathscr{C}}$ is covariant as seen from (5.6). From the bijection in

(5.6), the functor $\widetilde{\ }$ is fully faithful. And for any two objects X and X' in $\mathscr{C}$, if $\widetilde{X} = \widetilde{X}'$ in $\hat{\mathscr{C}}$, we must have $\widetilde{X}Y = \widetilde{X}'Y$ for any object Y of $\mathscr{C}$. Then $\mathrm{Hom}_{\mathscr{C}}(Y, X) = \mathrm{Hom}_{\mathscr{C}}(Y, X')$ implies $X = X'$ by Definition 1 of a category. Namely, $\widetilde{\ }$ is an embedding. The functor

$$\widetilde{\ } : \mathscr{C} \rightsquigarrow \hat{\mathscr{C}}$$

is called the *Yoneda embedding*.

Remark 5. Consider the following diagram of categories and functors:

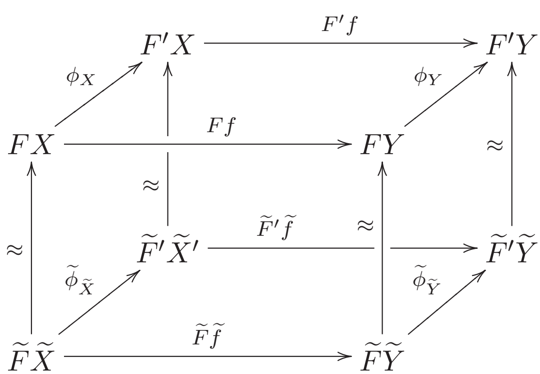


(5.7)

where $\widetilde{F} = \mathrm{Hom}_{\hat{\mathscr{C}}}(\cdot, F) : \hat{\mathscr{C}} \rightsquigarrow \mathsf{Set}$ is a contravariant functor. The commutativity of (5.7) is equivalent to the statement of Yoneda's Lemma (Proposition 4). If $\widetilde{F}$ is used, the Yoneda bijection (5.3) becomes the *lifting formula* of $(F, X) \in \hat{\mathscr{C}} \times \mathscr{C}$ to $(\widetilde{F}, \widetilde{X}) \in \hat{\hat{\mathscr{C}}} \times \hat{\mathscr{C}}$:

$$\widetilde{F}\widetilde{X} \approx FX. \tag{5.8}$$

Then for $f : Y \to X$ in $\mathscr{C}$, $\phi : F \to F'$ in $\hat{\mathscr{C}}$ and $\widetilde{\phi} : \widetilde{F} \to \widetilde{F}'$ in $\hat{\hat{\mathscr{C}}}$ we have the commutative diagram in Set:



(5.9)

where all the vertical morphisms are Yoneda's isomorphisms (bijections) in Set.

Notice also that $\sim(\mathscr{C}) := \{\widetilde{X} \mid X \in \mathrm{Ob}(\mathscr{C})\}$ forms a subcategory of $\hat{\mathscr{C}}$.

1.6 Abelian Categories

In the category Ab of abelian groups, for a group G consisting of one element $\mathsf{G} = \{0_{\mathsf{G}}\}$, there is only one morphism in $\mathrm{Hom}_{\mathsf{Ab}}(\mathsf{G}', \mathsf{G})$ for each $\mathsf{G}' \in \mathrm{Ob}(\mathsf{Ab})$.

In the category Set, a set of one element plays the same role. Namely, in general, for a category $\mathscr{C}$, an object Z of $\mathscr{C}$ is said to be a *terminal object* if the set $\mathrm{Hom}_{\mathscr{C}}(X, Z)$ has exactly one element for each X. An object A is said to be an *initial object* if the set $\mathrm{Hom}_{\mathscr{C}}(A, X)$ has exactly one element for every $X \in \mathrm{Ob}(\mathscr{C})$. An object 0 of $\mathscr{C}$ is said to be a *zero object* for $\mathscr{C}$ if 0 is both terminal and initial. Notice that for terminal objects Z and Z' in $\mathscr{C}$ we have $f^Z_{Z'} : Z \to Z'$ and $f^{Z'}_Z : Z' \to Z$ and we have $1_Z : Z \to Z$ and $1_{Z'} : Z' \to Z'$. Then since $\mathrm{Hom}_{\mathscr{C}}(Z, Z)$ has only one element $f^{Z'}_Z \circ f^Z_{Z'} = 1_Z$ and similarly we have $f^Z_{Z'} \circ f^{Z'}_Z = 1_{Z'}$. Consequently, for any terminal object $f^Z_{Z'} : Z \to Z'$ is an isomorphism in $\mathscr{C}$. The same is true for an initial and a zero object of a category. For any objects X and Y in $\mathscr{C}$, we have $f^X_0 : X \to 0$ and $g^0_Y : 0 \to Y$ obtaining $g^0_Y \circ f^X_0 : X \to Y$. This uniquely determined morphism $0^X_Y := g^0_Y \circ f^X_0$ is said to be a *zero morphism*. But in Remark 1 we have used the notion of a zero morphism to define the notion of a kernel.

A category $\mathscr{A}$ is said to be an *abelian category* if the following (Ab.1) through (Ab.6) are satisfied.

(Ab.1) For any X and Y in $\mathscr{A}$, $\mathrm{Hom}_{\mathscr{A}}(X, Y)$ is an object in Ab, i.e., an abelian group with respect to a binary composition $+_{X,Y}$ on the set $\mathrm{Hom}_{\mathscr{A}}(X, Y)$. Namely, for objects X, X', Y, Y' of $\mathscr{A}$ and morphisms given as

$$X' \xrightarrow{h} X \underset{g}{\overset{f}{\rightrightarrows}} Y \xrightarrow{k} Y' \tag{6.1}$$

we have $k \circ (f + g) = k \circ f + k \circ g$ in $\mathrm{Hom}_{\mathscr{A}}(X, Y')$ and

$$(f + g) \circ h = f \circ h + g \circ h$$

in $\mathrm{Hom}_{\mathscr{A}}(X', Y)$.

(Ab.2) A zero object 0 exists in $\mathscr{A}$. Then $\mathrm{Hom}_{\mathscr{A}}(0, 0)$ is the trivial abelian group.

(Ab.3) For any objects X and Y in $\mathscr{A}$ the direct sum (coproduct) $X \oplus Y$ exists in $\mathscr{A}$. That is, $X \oplus Y$ is an object in $\mathscr{A}$ which is representing the following covariant functor from $\mathscr{A}$ to Ab:

$$\mathrm{Hom}_{\mathscr{A}}(X, \cdot) \times \mathrm{Hom}_{\mathscr{A}}(Y, \cdot) : \mathscr{A} \rightsquigarrow \mathsf{Ab}. \tag{6.2}$$

Namely, for an object Z in $\mathscr{A}$, there is an isomorphism

$$\widetilde{X \oplus Y} Z := \mathrm{Hom}_{\mathscr{A}}(X \oplus Y, Z) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{A}}(X, Z) \times \mathrm{Hom}_{\mathscr{A}}(Y, Z). \tag{6.3}$$

(Ab.4) For a morphism $f : X \to Y$ in $\mathscr{A}$, the object $\ker f$ exists in $\mathscr{A}$. We have already mentioned the kernel of a morphism in Remark 1. Here is a definition of a kernel. The kernel $\ker f$ of a morphism $f : X \to Y$ is an object which represents the following contravariant functor:

$$\ker(\mathrm{Hom}_{\mathscr{A}}(\cdot, X) \to \mathrm{Hom}_{\mathscr{A}}(\cdot, Y)) : \mathscr{A} \rightsquigarrow \mathsf{Ab}. \tag{6.4}$$

Namely,

$$\widetilde{\ker f} Z := \mathrm{Hom}_{\mathscr{A}}(Z, \ker f) \xrightarrow{\approx} \ker(\mathrm{Hom}_{\mathscr{A}}(Z, X) \to \mathrm{Hom}_{\mathscr{A}}(Z, Y)). \tag{6.5}$$

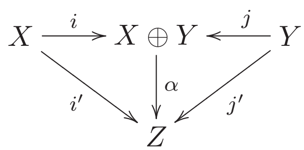
(Ab.5) For a morphism $f : X \to Y$, the object $\operatorname{coker} f$ exists in $\mathscr{A}$. Consider the functor

$$\ker(\mathrm{Hom}_{\mathscr{A}}(Y, \cdot) \to \mathrm{Hom}_{\mathscr{A}}(X, \cdot)) : \mathscr{A} \rightsquigarrow \mathsf{Ab}. \tag{6.6}$$

Then (6.6) is represented by the object $\operatorname{coker} f$:

$$\widetilde{\operatorname{coker} f} Z := \mathrm{Hom}_{\mathscr{A}}(\operatorname{coker} f, Z) \xrightarrow{\approx} \ker(\mathrm{Hom}_{\mathscr{A}}(Y, Z) \to \mathrm{Hom}_{\mathscr{A}}(X, Z)). \tag{6.7}$$

Remark 6. Before we mention the last condition for a category to be an abelian category, let us recall a few universal mapping properties for the notions that appeared in (Ab.3)–(Ab.5). The direct sum of X and Y is a pair of morphisms $i : X \to X \oplus Y$ and $j : Y \to X \oplus Y$ satisfying the following universal property. Namely, for each pair of morphisms $i' : X \to Z$ and $j' : Y \to Z$ there is a unique morphism $\alpha : X \oplus Y \to Z$ making the diagram



commutative, i.e., (6.3) in (Ab.3). Another example may be an element of the right hand-side of (6.5). That is, if $g : Z \to X$ satisfies $f \circ g = 0$, then there is a unique $h : Z \to \ker f$ satisfying $g = i \circ h$ where $i : \ker f \to X$ as in Remark 1. Namely, (6.3), (6.5) and (6.7) are exactly the universal mapping properties of the direct sum, the kernel and cokernel, respectively.

Now we return to the last condition (Ab.6). First notice that $\ker f \xrightarrow{i} X$ is a monomorphism. This is because: if $\phi, \psi : K \to \ker f$ satisfy $i \circ \phi = i \circ \psi$ from K to X then composing with $f : X \to Y$ we get $f \circ i \circ \phi = f \circ i \circ \psi = 0$ from K to Y. By the universal property of $\ker f \xrightarrow{i} X$ or by (6.5), there is a unique $\iota : K \to \ker f$ satisfying $i \circ \iota = i \circ \phi = i \circ \psi : K \to X$ concluding that $\iota = \phi = \psi$. Consequently $i : \ker f \to X$ is a monomorphism. By (Ab.5) $\operatorname{coker} i$ exists in $\mathscr{A}$. Define the coimage of $f : X \to Y$ as the cokernel of $i : \ker f \to X$, i.e., $\operatorname{coim} f := \operatorname{coker} i$. Next, let $Z = \operatorname{coker} f$

in (6.7). Then $1_{\operatorname{coker} f} \in \operatorname{Hom}_{\mathscr{A}}(\operatorname{coker} f, \operatorname{coker} f)$ determines the element $\pi \in \operatorname{Hom}_{\mathscr{A}}(Y, \operatorname{coker} f)$ satisfying $\pi \circ f = 0$. We define $\operatorname{im} f := \ker \pi$. The universal property for $\ker \pi$ or (6.5) implies that there is a unique morphism $g : X \to \ker \pi = \operatorname{im} f$ making the following diagram commutative.

$$\begin{array}{ccccccc} \ker f & \xrightarrow{i} & X & \xrightarrow{f} & Y & \xrightarrow{\pi} & \operatorname{coker} f \\ & & \downarrow{\scriptstyle \pi'} & \searrow^{g} & \uparrow{\scriptstyle i'} & & \\ & & \operatorname{coker} i & \xrightarrow{h} & \ker \pi & & \\ & & \| & & \| & & \\ & & \operatorname{coim} f & & \operatorname{im} f & & \end{array} \tag{6.8}$$

Furthermore, by the universality for $\operatorname{coker} i$, $g \circ i = 0$ implies that there is a unique morphism $h : \operatorname{coker} i \to \ker \pi = \operatorname{im} f$ making the above diagram commutative. Define $\operatorname{coim} f := \operatorname{coker} i$.

A category $\mathscr{A}$ satisfying (Ab.1)–(Ab.5) is said to be an *abelian category* if the factorization morphism

(Ab.6) $h : \operatorname{coim} f \to \operatorname{im} f$ is an isomorphism. Note that such an h is uniquely determined. This is because for another $h' : \operatorname{coim} f \to \operatorname{im} f$, the equality $i' \circ h \circ \pi' = i' \circ h' \circ \pi' = f$ implies $h \circ \pi' = h' \circ \pi'$ since i' is a monomorphism Then, since π' is an epimorphism, we get $h = h'$.

Note 6. When $\mathscr{A}$ is an abelian category, the opposite category as defined in 1.2, $\mathscr{A}^\circ$ is also abelian. This is because the dual statement of (Ab.2) is the same as (Ab.2), the dual object of the direct sum, which is called the *direct product*, is isomorphic to the direct sum, and (Ab.4)–(Ab.6) are dual to each other. We introduced the category $\mathscr{C}'^{\mathscr{C}}$ in (2.5) whose objects are functors from $\mathscr{C}$ to $\mathscr{C}'$ and morphisms are natural transformations of functors. If $\mathscr{C}'$ is an abelian category and if $\mathscr{C}$ is a small category (i.e., if $\operatorname{Ob}(\mathscr{C})$ is a set), then $\mathscr{C}'^{\mathscr{C}}$ inherits the property of being abelian from $\mathscr{C}'$. For an abelian category $\mathscr{A}$, the category $\mathsf{Co}(\mathscr{A})$ of cochain complexes becomes an abelian category. A definition of the category $\mathsf{Co}(\mathscr{A})$ will be given in Chapter II.

1.6.1 Embeddings of Abelian Categories

First recall from (4.1) that for a functor $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ we have the map

$$\bar{F} : \operatorname{Hom}_{\mathscr{C}}(X, Y) \to \operatorname{Hom}_{\mathscr{C}'}(FX, FY). \tag{6.9}$$

If $\mathscr{C}$ and $\mathscr{C}'$ are abelian categories, for $f, g \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$, we have that $f + g \in \operatorname{Hom}_{\mathscr{C}}(X, Y)$ and $Ff + Fg \in \operatorname{Hom}_{\mathscr{C}'}(FX, FY)$. Then $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ is said to be an *additive functor* if $\bar{F}$ is a group homomorphism. Namely, in $\operatorname{Hom}_{\mathscr{C}'}(FX, FY)$

$$\bar{F}(f + g) = \bar{F}f + \bar{F}g, \tag{6.10}$$

i.e., in $\mathscr{C}'$ we have $F(f+g) = Ff + Fg$.

The Embedding Theorem now states the following: there is a functor $'$ from a small abelian category $\mathscr{A}$ to the category Ab of abelian groups. Then $' : \mathscr{A} \rightsquigarrow \mathsf{Ab}$ is an additive functor and for an exact sequence

$$\cdots \longrightarrow X_{i-1} \longrightarrow X_i \longrightarrow X_{i+1} \longrightarrow \cdots$$

in $\mathscr{A}$ the sequence

$$\cdots \longrightarrow X'_{i-1} \longrightarrow X'_i \longrightarrow X'_{i+1} \longrightarrow \cdots$$

is exact in Ab. See Lubkin, S., *Imbedding of Abelian Categories*, Trans. Amer. Math. Soc. **97** (1960), pp. 410–417, for a proof. Consequently, this embedding theorem implies

(i) for an object X in $\mathscr{A}$ its image X' is an abelian group,

(ii) the image Y' of a subobject Y of X is a subgroup of X',

(iii) for a morphism $X \xrightarrow{f} Z$ in $\mathscr{A}$ the $\ker f$, $\operatorname{coker} f$, $\operatorname{im} f$ and $\operatorname{coim} f$ are identified with $\ker f'$, $\operatorname{coker} f'$, $\operatorname{im} f'$ and $\operatorname{coim} f'$ of $X' \xrightarrow{f'} Z'$ in Ab.

Moreover, the identification of $\mathscr{A}$ with the subcategory $\mathscr{A}' = {}'(\mathscr{A})$, diagram chasing in terms of elements may be carried out in Ab for a diagram in an abelian category. Recall that we have the Yoneda embedding $\widetilde{\ } : \mathscr{A} \rightsquigarrow \hat{\mathscr{A}} = \mathsf{Set}^{\mathscr{A}^\circ}$ defined by $X \mapsto \operatorname{Hom}_{\mathscr{A}}(\cdot, X) = \widetilde{X} \in \operatorname{Ob}(\hat{\mathscr{A}})$. A category is said to be *additive* if (Ab.1)–(Ab.3) are satisfied. For an additive category $\mathscr{A}$ and an additive functor $F : \mathscr{A} \rightsquigarrow \mathsf{Ab}$, Yoneda's lemma states that

$$\operatorname{Hom}_{\hat{\mathscr{A}}}(\widetilde{X}, F) \xrightarrow{\approx} FX \tag{6.11}$$

is a group isomorphism. Let us revisit (Ab.4). First, recall that

$$\widetilde{\ } : \mathscr{A} \rightsquigarrow \hat{\mathscr{A}} = \mathsf{Set}^{\mathscr{A}^\circ}$$

is a covariant functor. For $X \xrightarrow{f} Y$ in $\mathscr{A}$ we have

$$\widetilde{X} = \operatorname{Hom}_{\mathscr{A}}(\cdot, X) \xrightarrow{\tilde{f}} \widetilde{Y} = \operatorname{Hom}_{\mathscr{A}}(\cdot, Y)$$

in $\mathsf{Ab}^{\mathscr{A}^\circ} = \hat{\mathscr{A}}$. Since $\hat{\mathscr{A}}$ is abelian for an abelian category $\mathscr{A}$ the kernel of $\tilde{f}$ exists in $\hat{\mathscr{A}}$. Namely, (6.5) may be read as

$$\widetilde{\ker f} = \ker \tilde{f} \tag{6.12}$$

in $\hat{\mathscr{A}} = \mathsf{Ab}^{\mathscr{A}^\circ}$.

Remark 7. Let $\mathscr{A}$ be an abelian category and let

$$\cdots \longrightarrow X_{i-1} \xrightarrow{d_{i-1}} X_i \xrightarrow{d_i} X_{i+1} \xrightarrow{d_{i+1}} \cdots \tag{6.13}$$

be a sequence of objects and morphisms in $\mathscr{A}$. Then (6.13) is said to be an *exact sequence* if $\ker d_i = \operatorname{im} d_{i-1}$. If one prefers to regard (6.13) as a sequence in Ab, the equality $\ker d_i = \operatorname{im} d_{i-1}$ is set-theoretic. Moreover

$$0 \longrightarrow X' \xrightarrow{d'} X \xrightarrow{d} X'' \longrightarrow 0$$

is exact in $\mathscr{A}$ if and only if d' is a monomorphism, d is an epimorphism and $\ker d = \operatorname{im} d'$. Then

$$0 \longrightarrow X' \xrightarrow{d'} X \xrightarrow{d} X'' \longrightarrow 0$$

is said to be a *short exact sequence*. Let $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ be a covariant (or contravariant) functor of abelian categories. For an exact sequence

$$0 \longrightarrow \ker f \xrightarrow{i} X \xrightarrow{f} Y$$

if

$$0 \longrightarrow F \ker f \xrightarrow{Fi} FX \xrightarrow{Ff} FY$$

is exact in $\mathscr{B}$, i.e., $F \ker f = \ker Ff$, F is said to be a *kernel preserving functor*. Notice that F is kernel preserving if and only if F is a *left exact functor* in the following sense: for every short exact sequence

$$0 \longrightarrow X' \xrightarrow{d'} X \xrightarrow{d} X'' \longrightarrow 0$$

in $\mathscr{A}$,

$$0 \longrightarrow FX' \xrightarrow{Fd'} FX \xrightarrow{Fd} FX''$$

is exact in $\mathscr{B}$.

Remark 8. In (6.12) the equality $\widetilde{\ker f} = \ker \tilde{f}$ in $\hat{\mathscr{A}}$ implies that $\tilde{}\ : \mathscr{A} \rightsquigarrow \hat{\mathscr{A}}$ is a kernel preserving functor. Namely for

$$0 \longrightarrow \ker f \xrightarrow{i} X \xrightarrow{f} Y$$

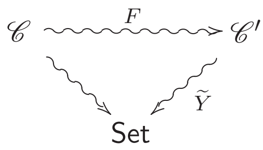
in $\mathscr{A}$ we have

$$0 \longrightarrow \ker \tilde{f} \xrightarrow{\tilde{i}} \widetilde{X} \xrightarrow{\tilde{f}} \widetilde{Y}\ .$$

That is, the Yoneda embedding $\tilde{}\ : \mathscr{A} \rightsquigarrow \hat{\mathscr{A}}$ is a left exact covariant functor which takes an object $X \in \mathrm{Ob}(\mathscr{A})$ to a left exact contravariant (or, covariant) functor $\widetilde{X} = \mathrm{Hom}_{\mathscr{A}}(\cdot, X)$ (or, $\widetilde{X} = \mathrm{Hom}_{\mathscr{A}}(X, \cdot)$) from the abelian category $\mathscr{A}$ to Ab.

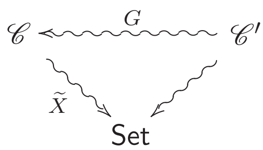
1.7 Adjoint Functors

Let $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ and $G : \mathscr{C}' \rightsquigarrow \mathscr{C}$ be functors. Consider the following diagrams:



(7.1)

and



(7.2)

Let Y be an arbitrary object of $\mathscr{C}'$. Then by the functor $\widetilde{\ } : \mathscr{C}' \rightsquigarrow \hat{\mathscr{C}}' = \mathsf{Set}^{\mathscr{C}'}$, $\widetilde{Y}$ is an object of $\hat{\mathscr{C}}'$, i.e., $\widetilde{Y}$ is a functor from $\mathscr{C}'$ to Set defined by $\widetilde{Y} = \mathrm{Hom}_{\mathscr{C}'}(\cdot, Y)$. (If the reader chooses to review some material for this discussion we suggest Section 1.5 to Remark 5.) Then the composition $\widetilde{Y}F$ is an object of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}}$. If there exists an object X' in $\mathscr{C}$ representing

$$\widetilde{Y}F : \mathscr{C} \rightsquigarrow \mathsf{Set}, \tag{7.3}$$

we get an isomorphism (called a *natural equivalence*)

$$\widetilde{X}' \xrightarrow{\approx} \widetilde{Y}F, \tag{7.4}$$

making (7.1) commutative. Moreover, if this representing object X' happens to be the image of Y under the functor G from $\mathscr{C}'$ to $\mathscr{C}$, i.e.,

$$\widetilde{GY} \xrightarrow{\approx} \widetilde{Y}F \tag{7.5}$$

in $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}}$, then G is said to be the *(right) adjoint* to F and F is said to be the *(left) adjoint* to G. Let us rewrite (7.5) as

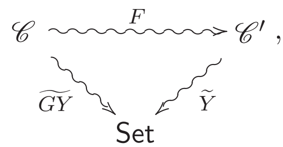
$$\mathrm{Hom}_{\mathscr{C}}(\cdot, GY) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}'}(F\,\cdot, Y) \tag{7.6}$$

in $\hat{\mathscr{C}}$. Namely, for every object X of $\mathscr{C}$ and for every object Y of $\mathscr{C}'$

$$\mathrm{Hom}_{\mathscr{C}}(X, GY) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}'}(FX, Y) \tag{7.7}$$

in Set.

Remark 9. Let $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ be adjoint to $G : \mathscr{C}' \rightsquigarrow \mathscr{C}$. Then from the commutative diagram as in (7.1) we have



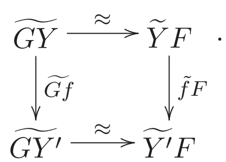
(7.8)

i.e., $\widetilde{GY} \approx \widetilde{Y}F$ in $\hat{\mathscr{C}}$. Let $Y = FX$ in (7.8). We get $\widetilde{GFX} \approx \widetilde{FX}F$. This is nothing but the substitution $Y = FX$ in (7.7), obtaining

$$\mathrm{Hom}_{\mathscr{C}}(X, GFX) \approx \mathrm{Hom}_{\mathscr{C}'}(FX, FX).$$

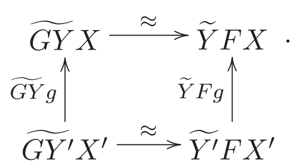
The identity 1_{FX} determines a morphism from X to GFX in $\mathscr{C}$. Namely, $1_F \in \mathrm{Ob}(\mathscr{C}'^{\mathscr{C}})$ determines the natural transformation $\alpha : 1_{\mathscr{C}} \to GF$ in $\mathscr{C}^{\mathscr{C}}$. Similarly, evaluate $\widetilde{GY} \approx \widetilde{Y}F$ at $X = GY$, i.e., substituting $X = GY$ in (7.7), to obtain $\mathrm{Hom}_{\mathscr{C}}(GY, GY) \approx \mathrm{Hom}_{\mathscr{C}'}(FGY, Y)$. Then 1_{GY} determines $\beta_Y : FGY \to Y$ in $\mathscr{C}'$ inducing $\beta : FG \to 1_{\mathscr{C}'}$.

Moreover, for $Y \xrightarrow{f} Y'$ in $\mathscr{C}'$ we have the following diagram in $\hat{\mathscr{C}}$:



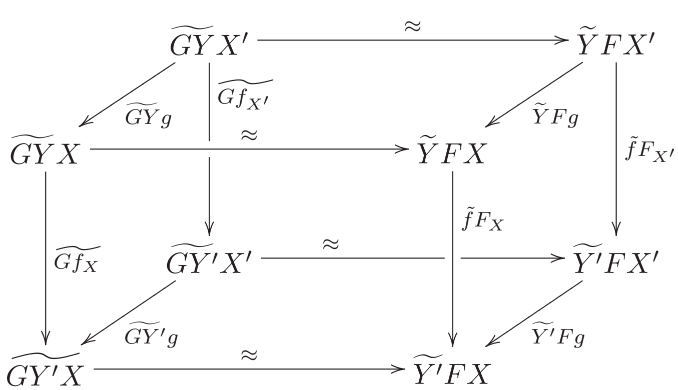
(7.9)

And for $X \xrightarrow{g} X'$ in $\mathscr{C}$, we have the diagram in Set



(7.10)

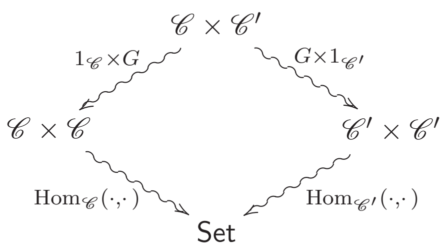
Diagrams (7.9) and (7.10) may be combined in Set as



(7.11)

Remark 10. We can also express an adjoint pair

as follows. Let $1_{\mathscr{C}} : \mathscr{C} \rightsquigarrow \mathscr{C}$ and $1_{\mathscr{C}'} : \mathscr{C}' \rightsquigarrow \mathscr{C}'$ be identity functors of $\mathscr{C}$ and $\mathscr{C}'$, respectively. The functor $G : \mathscr{C}' \rightsquigarrow \mathscr{C}$ is said to be the (right) adjoint to the functor $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ when the following diagram of categories and functors commute.



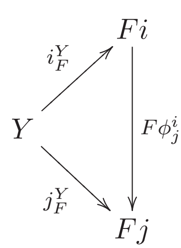
(7.12)

Actually, as noted in (7.7), there is a natural equivalence from the composition of $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$ and $1_{\mathscr{C}} \times G$ to the composition of $\mathrm{Hom}_{\mathscr{C}'}(\cdot, \cdot)$ and $F \times 1_{\mathscr{C}'}$ in (7.12). Note that $\mathscr{C} \times \mathscr{C}'$ is the product category of $\mathscr{C}$ and $\mathscr{C}'$ whose objects are ordered pairs (A, A') with $A \in \mathrm{Ob}(\mathscr{C})$ and $A' \in \mathrm{Ob}(\mathscr{C}')$. The set of morphisms $\mathrm{Hom}_{\mathscr{C} \times \mathscr{C}'}((A, A'), (B, B'))$ is the product set $\mathrm{Hom}_{\mathscr{C}}(A, B) \times \mathrm{Hom}_{\mathscr{C}'}(A', B')$. The functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, \cdot)$ is called a bifunctor from $\mathscr{C} \times \mathscr{C}$ to Set defined by

$$(A, B) \in \mathrm{Ob}(\mathscr{C} \times \mathscr{C}) \mapsto \mathrm{Hom}_{\mathscr{C}}(A, B) \in \mathrm{Ob}(\mathsf{Set}).$$

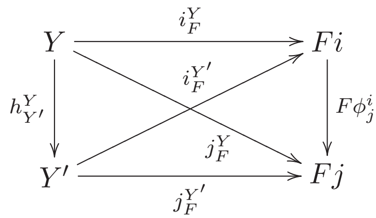
1.8 Limits

Let $\mathscr{C}$ be $\mathscr{C}'$ be categories and let F be a (covariant) functor from $\mathscr{C}'$ to $\mathscr{C}$. Then we will define the category $\mathfrak{F}n^F$ *of (left) fans with fixed objects* with respect to F. An object of $\mathfrak{F}n^F$ is (Y, i_F^Y, Fi), where Y is an object of $\mathscr{C}$ and i, j are objects in $\mathscr{C}'$ making the triangle



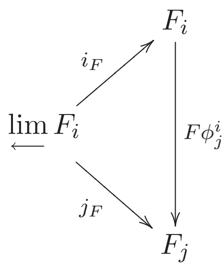
(8.1)

commutative, for $\phi_j^i : i \to j$ in $\mathscr{C}'$. A morphism from $(Y, i_F^Y, Fi)_{i\in\mathscr{C}'}$ to $(Y', i_F^{Y'}, Fi)_{i\in\mathscr{C}'}$ is defined as $h_{Y'}^Y : Y \to Y'$ making



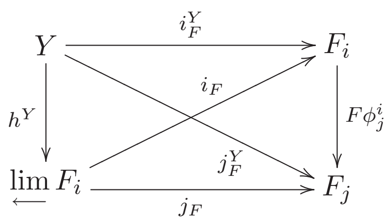
(8.2)

commutative, i.e., $i_F^Y = i_F^{Y'} \circ h_{Y'}^Y$ and $j_F^Y = j_F^{Y'} \circ h_{Y'}^Y$ for $\phi_j^i : i \to j$. A terminal object of $\mathfrak{F}n^F$ is said to be an *inverse limit* (or *projective limit* or simply *limit*) of F written as $\varprojlim_{i\in\mathscr{C}'} Fi$, or $\varprojlim F_i$. Namely, $\varprojlim F_i \in \mathrm{Ob}(\mathfrak{F}n^F)$:



(8.3)

commutes and for any object (Y, i_F^Y, Fi) in $\mathfrak{F}n^F$, there exists a unique morphism $h^Y : Y \to \varprojlim F_i$ making



(8.4)

commutative.

There is another way to express (8.1) through (8.4) in terms of the notion of a representable functor. First, we will define a functor $\iota : \mathscr{C} \rightsquigarrow \mathscr{C}^{\mathscr{C}'}$ as follows. Let $Y \xrightarrow{f} Y'$ be a morphism of objects Y and Y' in $\mathscr{C}$. Then $\iota Y \xrightarrow{\iota f} \iota Y'$ are in $\mathscr{C}^{\mathscr{C}'}$. For $i \in \mathrm{Ob}(\mathscr{C}')$ define $(\iota Y)(i) \xrightarrow{(\iota f)i} (\iota Y')(i)$ as $Y \xrightarrow{f} Y'$ in $\mathscr{C}$, i.e., $(\iota Y)(i) = Y$ and $(\iota f)(i) = f$ for every $i \in \mathrm{Ob}(\mathscr{C}')$. Let $F : \mathscr{C}' \rightsquigarrow \mathscr{C}$ be a functor as before. We can consider the set $\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota Y, F) \in \mathrm{Ob}(\mathsf{Set})$. That is,

$$\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota\,\cdot, F) : \mathscr{C} \rightsquigarrow \mathsf{Set} \tag{8.5}$$

is a contravariant functor. Then a representing object for this functor (8.5) is an inverse limit for F. Namely, there is an object $\varprojlim F_i$ in $\mathscr{C}$ such that as objects in $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}}$

$$\widetilde{\varprojlim F_i} \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota\,\cdot, F) \tag{8.6}$$

is an isomorphism (a natural equivalence). As objects in Set

$$\mathrm{Hom}_{\mathscr{C}}(Y, \varprojlim F_i) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota Y, F) \tag{8.7}$$

is an isomorphism for every objects Y of $\mathscr{C}$.

Incidentally, the functor $\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota\,\cdot, F)$ in (8.5) may be interpreted as the composition of functors, i.e., $\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota\,\cdot, F) = \widetilde{F} \circ \iota$ as in Section 1.7 (7.8). See the diagram

$$\begin{array}{ccc} \mathscr{C} & \overset{\iota}{\rightsquigarrow} & \mathscr{C}^{\mathscr{C}'} \\ & {\scriptstyle \widetilde{F}\circ\iota}\searrow & \Big\downarrow {\scriptstyle \widetilde{F}=\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\cdot, F)} \\ & & \mathsf{Set} \end{array} \tag{8.8}$$

Then an inverse limit $\varprojlim F_i$ is an object of $\mathscr{C}$ which represents the composition $\widetilde{F} \circ \iota = \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota\,\cdot, F)$ of ι followed by $\widetilde{F}$ in (8.8).

Note 7. Let us observe that (8.7) implies (8.3) and (8.4). In (8.7) let $Y = \varprojlim F_i$, i.e.,

$$\mathrm{Hom}_{\mathscr{C}}(\varprojlim F_i, \varprojlim F_i) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota \varprojlim F_i, F).$$

For an identity morphism $1_{\varprojlim F_i}$ on the left hand-side, there is

$$\alpha \in \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota \varprojlim F_i, F).$$

For this natural transformation $\alpha : \iota \varprojlim F_i \to F$, compute at $i \xrightarrow{\phi_j^i} j$ in $\mathscr{C}'$ as follows

$$\begin{array}{ccc} (\iota \varprojlim F_i)i := \varprojlim F_i & \xrightarrow{\alpha_i} & Fi = F_i \\ {\scriptstyle 1_{\varprojlim F_i}}\Big\downarrow & & \Big\downarrow{\scriptstyle F\phi_j^i} \\ (\iota \varprojlim F_i)j := \varprojlim F_i & \xrightarrow{\alpha_j} & Fj = F_j \end{array} \tag{8.9}$$

which is (8.3). Next let $-^Y_F : \iota Y \to F$ be a morphism in $\mathscr{C}^{\mathscr{C}'}$. For $i \xrightarrow{\phi^i_j} j$ compute i^Y_F and j^Y_F as

$$\begin{array}{ccc} (\iota Y)i = Y & \xrightarrow{i^Y_F} & Fi = F_i \\ {\scriptstyle 1_Y}\downarrow & & \downarrow{\scriptstyle F\phi^i_j} \\ (\iota Y)j = Y & \xrightarrow{j^Y_F} & Fj = F_j \end{array} \tag{8.10}$$

For this element $-^Y_F \in \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(\iota Y, F)$ on the right hand-side of (8.7) there exists a unique element $h^Y \in \mathrm{Hom}_{\mathscr{C}}(Y, \varprojlim F_i)$. Then (8.9) and (8.10) give (8.4).

1.9 Dual Notion of Inverse Limit

Let $F : \mathscr{C}' \rightsquigarrow \mathscr{C}$ be a functor. Consider the following diagram corresponding to (8.8):

$$\begin{array}{ccc} \mathscr{C} & \overset{\iota}{\rightsquigarrow} & \mathscr{C}^{\mathscr{C}'} \\ & {\scriptstyle \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F,\iota\,\cdot)} \searrow & \downarrow{\scriptstyle \widetilde{F}:=\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F,\cdot)} \\ & & \mathsf{Set} \end{array} \tag{9.1}$$

Then a representing object in $\mathscr{C}$ for the composed covariant functor

$$\widetilde{F} \circ \iota = \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F, \iota\,\cdot)$$

from $\mathscr{C}$ to Set is the *direct limit* (or *colimit*) $\varinjlim F_i$ of F. Namely, we have the isomorphism of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}}$

$$\widetilde{\varinjlim F_i} \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F, \iota\,\cdot). \tag{9.2}$$

As objects of Set, for every $Y \in \mathrm{Ob}(\mathscr{C})$, we have

$$\widetilde{\varinjlim F_i} Y = \mathrm{Hom}_{\mathscr{C}}(\varinjlim F_i, Y) \xrightarrow{\approx} \mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F, \iota Y). \tag{9.3}$$

That is, for $\phi^i_j : i \to j$ in $\mathscr{C}'$ we have the commutative diagram

$$\begin{array}{ccc} F_i & & \\ & \searrow^{i^F} & \\ {\scriptstyle F\phi^i_j}\downarrow & & \varinjlim F_i \\ & \nearrow_{j^F} & \\ F_j & & \end{array} \tag{9.4}$$

and in the category $\mathfrak{F}n_F$ of (right) fans with fixed objects with respect to F, the object of (9.4) is an initial object. Namely, if

$$\begin{array}{ccc} F_i & & \\ {\scriptstyle F\phi^i_j}\downarrow & \searrow^{i^F_Y} & \\ & & Y \\ & \nearrow_{j^F_Y} & \\ F_j & & \end{array} \tag{9.5}$$

is an object of $\mathfrak{F}n_F$ (i.e., an element of $\mathrm{Hom}_{\mathscr{C}^{\mathscr{C}'}}(F, \iota Y)$, the right hand-side of (9.3)), then there is a unique morphism h_Y from $\varinjlim F_i$ to Y making the diagram

$$\begin{array}{ccc} F_i & \xrightarrow{i^F} & \varinjlim F_i \\ {\scriptstyle F\phi^i_j}\downarrow & \overset{j_F\ \nearrow}{\underset{\searrow\ i^F_Y}{\times}} & \downarrow{\scriptstyle h_Y} \\ F_j & \xrightarrow[j^F_Y]{} & Y \end{array} \tag{9.6}$$

commutative.

Note 8. For a functor $F : \mathscr{C}' \rightsquigarrow \mathscr{C}$, the definition of an inverse limit becomes the *direct product* $\prod_{i\in\mathscr{C}'} F_i$ if $\mathscr{C}'$ is a discrete category (that is, if $\mathscr{C}'$ has no morphisms except identities). Similarly, a *direct sum* $\bigoplus_{i\in\mathscr{C}'} F_i$ is a direct limit of F from a discrete category $\mathscr{C}'$ to $\mathscr{C}$.

1.10 Presheaves

In Subsection 1.2.1 we defined a presheaf F as a contravariant functor from the category $\mathscr{T}$ associated with a topological space T to the category Set or the category Ab. We will find it convenient to define the notion of a presheaf as an object of $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$ for any category $\mathscr{C}$. For example, for an object X of $\mathscr{C}$ the functor $\widetilde{X} = \mathrm{Hom}_{\mathscr{C}}(\cdot, X)$ is a presheaf over $\mathscr{C}$. In this section we will consider mostly the case $\hat{\mathscr{C}} = \hat{\mathscr{T}}$ with values in Set or Ab. Let $F \in \hat{\mathscr{T}} = \mathsf{Set}^{\mathscr{T}^\circ}$ and let $i : U \hookrightarrow V$ be an inclusion morphism. Then the induced map $Fi : FV \to FU$ is said to be the restriction map in Set. We often write FV as $F(V)$. Let U and V be objects in $\mathscr{T}$ (i.e., U and V are open sets in the topology for T). Then $i : U \cap V \hookrightarrow U$ and $j : U \cap V \hookrightarrow V$ are morphisms in $\mathscr{T}$. The restriction maps $F(U) \xrightarrow{Fi} F(U \cap V)$ and $F(V) \xrightarrow{Fj} F(U \cap V)$ are induced. We will give the definition of a presheaf explicitly as follows.

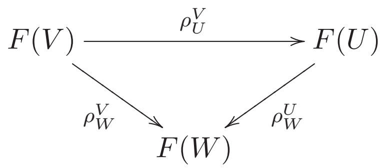
Definition 4. A functor $F : \mathscr{T} \rightsquigarrow \mathsf{Set}$ is a *presheaf* if (PreSh1)–(PreSh2′) are satisfied:

(PreSh1) For an open set U, $F(U)$ is a set where an element of $F(U)$ is said to be a *section* of F over U.

(PreSh2) For $U \subset V$ there is the induced map $\rho^V_U : F(V) \to F(U)$ called the *restriction map*. The following axioms must be satisfied:

(PreSh1′) For $U \in \mathrm{Ob}(\mathscr{T})$, ρ^U_U is the identity map $1_{F(U)} : F(U) \to F(U)$.

(PreSh2′) For open sets $W \subset U \subset V$ the diagram



commutes, i.e., $\rho^U_W \circ \rho^V_U = \rho^V_W$.

Note 9. Notice that all the conditions (PreSh1)–(PreSh2′) mean precisely that $F \in \mathrm{Ob}(\hat{\mathscr{T}})$.

Consider open sets $U, U', U'', \ldots$ containing a point x in the topological space T. Define an equivalence relation $\sim$ between $s \in F(U)$ and $s' \in F(U')$ as follows: $s \sim s'$ if and only if there is an open set V with $V \subset U \cap U'$ so that $\rho^U_V(s) = \rho^{U'}_V(s')$. The equivalence class s_x said to be the *germ* of $s \in F(U)$ (or $s' \in F(U')$) at x. The set F_x of all the germs at x is said to be the *stalk* of F at x. That is, for all open sets containing x, the direct limit

$$F_x = \varinjlim_{x \in U} F(U) \tag{10.1}$$

is the stalk of F at x.

Definition 5. A presheaf $F \in \mathrm{Ob}(\hat{\mathscr{T}})$ is said to be a *sheaf* when the following condition (Sheaf) is satisfied:

(Sheaf) Let U be an open set in T. For any open covering $\{U_i\}_{i \in I}$ of U (i.e., each U_i is an open set and $U = \bigcup_{i \in I} U_i$) and for any sections $\{s_i \in F(U_i)\}_{i \in I}$ satisfying

$$\rho^{U_i}_{U_i \cap U_j}(s_i) = \rho^{U_j}_{U_i \cap U_j}(s_j), \qquad \text{for} \quad i, j \in I \tag{10.2}$$

there exists a unique $s \in F(U)$ such that

$$\rho^U_{U_i}(s) = s_i, \qquad \text{for all } i \in I. \tag{10.3}$$

See the diagram below for (10.2) and (10.3).

$$\begin{array}{ccccc} & & F(U) & & \\ & \swarrow^{\rho^U_{U_i}} & & \searrow^{\rho^U_{U_j}} & \\ F(U_i) & & & & F(U_j) \\ & \searrow_{\rho^{U_i}_{U_i \cap U_j}} & & \swarrow_{\rho^{U_j}_{U_j \cap U_i}} & \\ & & F(U_i \cap U_j) & & \end{array} \tag{10.4}$$

Note 10. A presheaf

$$\mathcal{O}(U) = \{\text{holomorphic functions over } U \subset \mathbb{C}^n\}$$

is a sheaf. For a non-example consider $y = 1/x$, where x is a real number satisfying $0 < x < \infty$. Then

$$\mathcal{B}(U) = \{\text{locally bounded continuous functions on } U \subset (0, \infty)\}$$

is a presheaf but not a sheaf.

Remarks 1. (1) For the category $\hat{\mathscr{T}}$ of presheaves we let $\tilde{\mathscr{T}}$ be the category of sheaves over $\mathscr{T}$ (or over the topological space T).

(2) In Ab the following

$$A \xrightarrow{\alpha} B \underset{\beta'}{\overset{\beta}{\rightrightarrows}} C$$

is said to be exact if α is bijective onto the subset B' of B where

$$B' = \{b \in B \mid \beta(b) = \beta'(b)\}.$$

Then for $b \in B'$ there is a unique element $a \in A$ such that $\alpha(a) = b$. Consequently $(\beta \circ \alpha)(a) = (\beta' \circ \alpha)(a)$. Namely $A \xrightarrow{\alpha} B$ is the kernel of $\beta - \beta'$ (categorically speaking, α is an *equalizer* for β and β'). Then the sheaf axiom (Sheaf) may be summarized as the exact sequence

$$F(U) \xrightarrow{\rho^U_{U_i}} \prod_{i \in I} F(U_i) \underset{\rho^{U_j}_{U_j \cap U_i}}{\overset{\rho^{U_i}_{U_i \cap U_j}}{\rightrightarrows}} \prod_{i,j \in I} F(U_i \cap U_j)\ . \tag{10.5}$$

1.11 Notion of Site

The definitions of a presheaf and a sheaf have nothing to do with the elements in open sets in the category $\mathscr{T}$ induced from a topological space T. We will give much more general concepts of a sheaf and a presheaf over a category which will be called a site.

Let $\mathscr{C}$ be a category and let $\hat{\mathscr{C}} = \mathsf{Ab}^{\mathscr{C}^\circ}$ where as before Ab is the category of abelian groups. As already mentioned in Section 1.10 an object of $\hat{\mathscr{C}}$ is a presheaf with values in Ab. Let $U \in \mathrm{Ob}(\mathscr{C})$ and consider a collection $\mathrm{Cov}(U)$ of families of morphisms in $\mathscr{C}$. Each family of morphisms for an object U in $\mathscr{C}$

$$\{U_i \xrightarrow{f_i} U\}_{i \in I} \in \mathrm{Cov}(U)$$

is said to be a *covering family* of U when the following conditions are satisfied:

(Site1) An isomorphism $U' \xrightarrow{\approx} U$ is a covering family of U, i.e., the family of one morphism $\{U' \xrightarrow{\approx} U\} \in \mathrm{Cov}(U)$.

(Site2) Let $\{U_i \xrightarrow{f_i} U\}_{i \in I} \in \mathrm{Cov}(U)$. Then for $V \to U$ we have

$$_i\{U_i \times_U V \to V\} \in \mathrm{Cov}(V),$$

i.e., stable under a pullback. See the diagram below.

$$\begin{array}{ccc} U_i & \longrightarrow & U \\ \uparrow & & \uparrow \\ U_i \times_U V & \longrightarrow & V \end{array} \qquad (11.1)$$

(Site3) Let $\{U_i \xrightarrow{f_i} U\} \in \mathrm{Cov}(U)$ and $\{U_{ij} \xrightarrow{f'_j} U_i\} \in \mathrm{Cov}(U_i)$. Then the family of morphisms obtained by the compositions $\{U_{ij} \xrightarrow{f_i \circ f'_j} U\}$ belongs to $\mathrm{Cov}(U)$.

Then $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$, where $\mathrm{Cov}(\mathscr{C}) = \{\mathrm{Cov}(U) \mid U \in \mathrm{Ob}(\mathscr{C})\}$ is said to be a *site*. A *morphism* h of sites is a functor from $\mathscr{C}$ to $\mathscr{C}'$ satisfying: for $\{U_i \xrightarrow{f_i} U\} \in \mathrm{Cov}(U)$, we have $\{hU_i \xrightarrow{hf_i} hU\} \in \mathrm{Cov}(hU)$ (where $\mathrm{Cov}(hU)$ is an element of $\mathrm{Cov}(\mathscr{C}')$) and for $V \to U$, $h(U_i \times_U V) \to hU_i \times_{hU} hV$ is an isomorphism.

1.12 Sheaves over Site

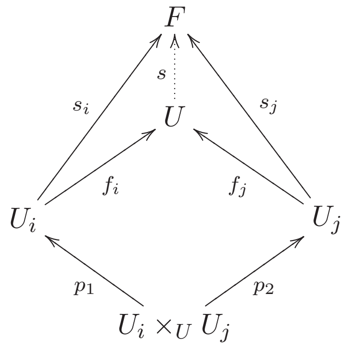
A presheaf $F \in \mathrm{Ob}(\hat{\mathscr{C}}) = \mathrm{Ob}(\mathsf{Ab}^{\mathscr{C}^\circ})$ is said to be a sheaf over $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$ if the diagram

$$F(U) \longrightarrow \prod F(U_i) \rightrightarrows \prod F(U_i \times_U U_j)$$

corresponding to (10.5) is exact. This full subcategory $\widetilde{\mathscr{C}}$ of sheaves of $\hat{\mathscr{C}}$ is said to be a *topos* over the site $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$. A morphism of sheaves is a morphism of presheaves. The above exact diagram may be written as

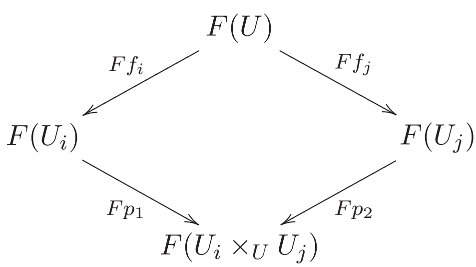
$$\mathrm{Hom}_{\hat{\mathscr{C}}}(\hat{U}, F) \longrightarrow \prod \mathrm{Hom}_{\hat{\mathscr{C}}}(\hat{U}_i, F) \rightrightarrows \prod \mathrm{Hom}_{\hat{\mathscr{C}}}(\widehat{U_i \times_U U_j}, F) \quad (12.1)$$

by Yoneda's Lemma. See the following diagrams below which correspond to (10.4).



(12.2a)

such that if $s_i \circ p_1 = s_j \circ p_2 \Rightarrow \exists ! s \in \mathrm{Hom}_{\hat{\mathscr{C}}}(U, F)$ satisfying $s_i = s \circ f_i$ for $i \in I$;



(12.2b)

1.13 Sieve; another notion for a site

Let $\mathscr{C}$ be a category, let $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$ be the category of presheaves and let $\widetilde{\ } : \mathscr{C} \rightsquigarrow \hat{\mathscr{C}}$ be the Yoneda embedding. Let $U \in \mathrm{Ob}(\mathscr{C})$. Then we are interested in a subobject of $\widetilde{U} = \mathrm{Hom}_{\mathscr{C}}(\cdot, U)$. Note that a subobject of a category is an equivalence class. (See Remark 1.) Let $i : \hat{W} \hookrightarrow \widetilde{U}$ be a subobject of $\widetilde{U}$ in $\hat{\mathscr{C}}$ where $\hat{W} \in \mathrm{Ob}(\hat{\mathscr{C}})$ need not be representable, i.e., $\hat{W}$ may not be replaced by $\widetilde{W}$ for some $W \in \mathrm{Ob}(\mathscr{C})$. Such a subobject $\hat{W}$ is said to be a *sieve of* $U \in \mathrm{Ob}(\mathscr{C})$. For $V \in \mathrm{Ob}(\mathscr{C})$ we have $\widetilde{U}V = \mathrm{Hom}_{\mathscr{C}}(V, U)$ and for a monomorphism representing the subobject $i : \hat{W} \hookrightarrow \widetilde{U}$, we have the set-theoretic inclusion $i_V : \hat{W}V \hookrightarrow \widetilde{U}V$. Namely, to give a sieve $\hat{W}$ of U is to determine a subset $\hat{W}V$ of $\mathrm{Hom}_{\mathscr{C}}(V, U)$ for every $V \in \mathrm{Ob}(\mathscr{C})$. By the Yoneda Lemma we have $\hat{W}V = \mathrm{Hom}_{\hat{\mathscr{C}}}(\widetilde{V}, \hat{W})$. The following is the Yoneda world diagram, identifying an object of $\mathscr{C}$ with the represented object of $\hat{\mathscr{C}}$:

$$\begin{array}{ccc} \hat{W} & \xrightarrow{i} & U \\ & \nwarrow^{\phi'} & \uparrow \phi \\ & & V \end{array} \quad \text{in } \hat{\mathscr{C}} \tag{13.1}$$

where $i \circ \phi' = i_V(\phi') = \phi$. A pair $(\mathscr{C}, \mathfrak{J}(\mathscr{C}))$ is said to be a *site*, where $\mathfrak{J}(\mathscr{C}) = \{\mathfrak{J}(U) \mid U \in \mathrm{Ob}(\mathscr{C})\}$ if each set $\mathfrak{J}(U)$ of sieves for U satisfies the following conditions.

(Site1$'$) An identity morphism $1_U : U \hookrightarrow U$ in $\hat{\mathscr{C}}$ is an element of $\mathfrak{J}(U)$.

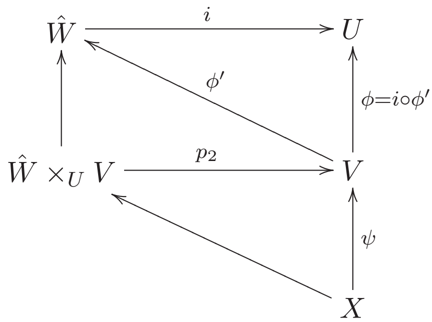
(Site2$'$) Let $\hat{W} \in \mathfrak{J}(U)$. Then for $V \xrightarrow{\phi} U$ in $\mathscr{C}$, $\hat{W} \times_U V \to V$ in $\hat{\mathscr{C}}$ belongs to the set $\mathfrak{J}(V)$ of sieves for V.

$$\begin{array}{ccc} \hat{W} & \overset{i}{\hookrightarrow} & U \\ \uparrow & & \uparrow \phi \\ \hat{W} \times_U V & \hookrightarrow & V \end{array} \tag{13.2}$$

(Site3$'$) Suppose $\hat{W} \in \mathfrak{J}(U)$ and let $\hat{W}' \hookrightarrow U$ be a sieve for U in $\hat{\mathscr{C}}$. For an arbitrary $V \in \mathrm{Ob}(\mathscr{C})$ and for every $V \xrightarrow{\phi'} \hat{W}$ in $\hat{\mathscr{C}}$, when the pullback of $\hat{W}'$ under $\phi = i \circ \phi'$, i.e., $\hat{W} \times_U V \to V$ is an element of $\mathfrak{J}(V)$, then $\hat{W}' \hookrightarrow U$ also belongs to $\mathfrak{J}(U)$.

$$\begin{array}{ccc} \hat{W} & \overset{i'}{\hookrightarrow} & U \\ \uparrow & & \uparrow {\scriptstyle \phi = i \circ \phi'} \\ \hat{W} \times_U V & \longrightarrow & V \end{array} \tag{13.3}$$

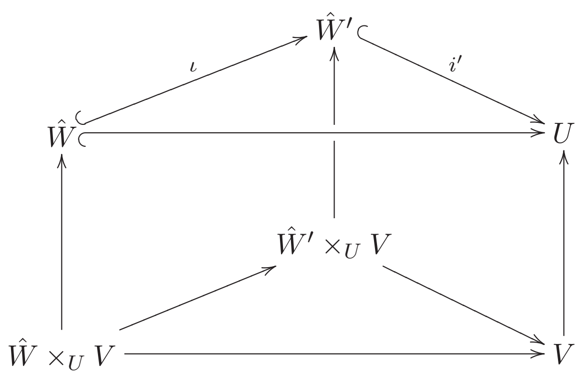
Remark 11. Those sieves belonging to $\mathfrak{J}(U)$ are said to be *covering sieves* for $U \in \mathrm{Ob}(\mathscr{C})$. Consider the case as in (13.1), a morphism $\phi : V \to U$ is factorable through a sieve $\hat{W}$, i.e., $\phi = i \circ \phi'$. Consider the following pullback diagram and an arbitrary morphism $\psi : X \to V$:



(13.4)

Then ψ can be factored as $\psi = p_2 \circ (\phi' \circ \psi, \psi)$ where p_2 is the projection onto the second factor. Namely, $VX \subset (\hat{W} \times_U V)X$ holds. Consequently $\hat{W} \times_U V = V$.

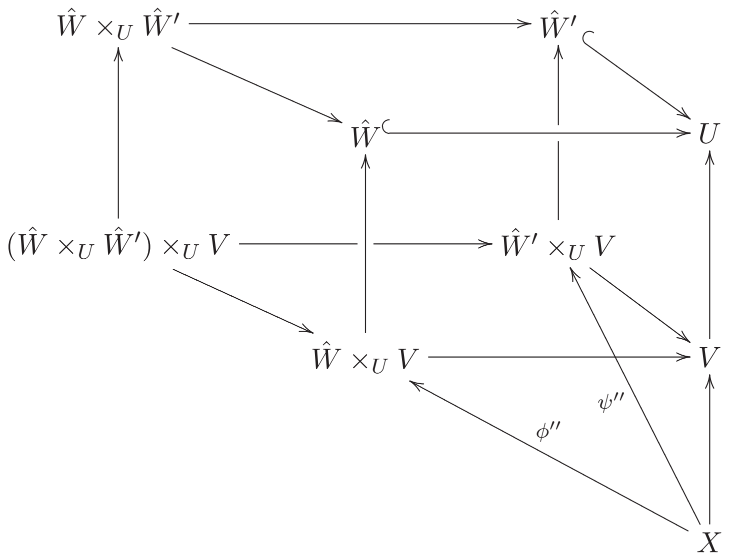
Remark 12. Let $\hat{W}$ and $\hat{W}'$ be sieves of U and $\hat{W}$ be a subobject of $\hat{W}'$ which is represented by a monomorphism $\iota : \hat{W} \hookrightarrow \hat{W}'$. If $\hat{W}$ is a covering sieve, i.e., $\hat{W} \in \mathfrak{J}(U)$, then so is $\hat{W}'$. A proof follows from the diagram:



(13.5)

That is, for an arbitrary morphism $\phi' : V \to \hat{W}$, the composition $\iota \circ \phi'$ is a morphism from V to $\hat{W}'$. Then from Remark 11 we have the pullbacks $\hat{W} \times_U V = V$ and $\hat{W}' \times_U V = V$. In particular, $\hat{W}' \times_U V = V \xrightarrow{1_V} V$ belongs to $\mathfrak{J}(V)$ by (Site1$'$). By (Site3$'$), $i' : \hat{W}' \hookrightarrow U$ is a covering sieve, i.e., $\hat{W}' \in \mathfrak{J}(U)$.

For covering sieves $\hat{W}$ and $\hat{W}'$ in $\mathfrak{J}(U)$, the pullback $\hat{W} \times_U \hat{W}'$ is a covering sieve. This follows from the following two-level pullback diagram:



(13.6)

Namely, for $\phi' : V \to \hat{W}$ as in Remark 11, a morphism from X to $\hat{W} \times_U V$ is induced. Then for each morphism ψ'' from X to $\hat{W}' \times_U V$ (and with $X \xrightarrow{\phi''} \hat{W} \times_U V$) there exists a unique morphism from X to $(\hat{W} \times_U \hat{W}') \times_U V$ giving commutativity. That is, each ψ'' can be factored through $(\hat{W} \times_U \hat{W}') \times_U V$. Consequently, $(\hat{W} \times_U \hat{W}') \times_U V = \hat{W}' \times_U V$ holds as in Remark 11. Since $\hat{W}' \in \mathfrak{J}(U)$, $\hat{W}' \times_U V$ is a covering sieve of V. Therefore, $(\hat{W} \times_U \hat{W}') \times_U V$ is a covering sieve of V. By (Site3′), $\hat{W} \times_U \hat{W}' \hookrightarrow U$ is covering sieve of U.

Note 11. Let $(\mathscr{C}, \mathrm{Cov}(\mathscr{C}))$ and $(\mathscr{C}, {}'\,\mathrm{Cov}(\mathscr{C}))$ be sites. Then ${}'\,\mathrm{Cov}(\mathscr{C})$ is said to be *finer* than $\mathrm{Cov}(\mathscr{C})$ if for each object U of $\mathscr{C}$, $\mathrm{Cov}(U) \subset {}'\,\mathrm{Cov}(U)$ holds.

Remark 13. Recall from Remark 1 that $V \xrightarrow{\phi} U$ is said to be an epimorphism when the contravariant functor $\mathrm{Hom}_{\mathscr{C}}(\cdot, W)$ always induces an injective map $\mathrm{Hom}_{\mathscr{C}}(U, W) \to \mathrm{Hom}_{\mathscr{C}}(V, W)$ in the category Set, i.e., $f \circ \phi = g \circ \phi$ in $\mathrm{Hom}_{\mathscr{C}}(V, W)$ implies $f = g$. A family of morphisms $\{U_i \xrightarrow{f_i} U\}_{i \in I}$ is said to be an *effective epimorphism* if for each object $W \in \mathrm{Ob}(\mathscr{C})$ the presheaf $\widetilde{W} = \mathrm{Hom}_{\mathscr{C}}(\cdot, W)$ satisfies the sheaf axiom for this family:

$$\widetilde{W}U \longrightarrow \prod_{i \in I} \widetilde{W}U_i \rightrightarrows \prod_{i,j \in I} \widetilde{W}(U_i \times_U U_j)$$

is exact in the sense of Remarks 1. Furthermore, a family of morphisms

$$\{U_i \xrightarrow{f_i} U\}_{i \in I}$$

is said to be *universal effective epimorphism* if for an arbitrary morphism $V \to U$ the family of pullback morphisms $\{U_i \times_U V \to V\}_{i \in I}$ is also an effective epimorphism. For a category $\mathscr{C}$ define

$$\overline{\mathrm{Cov}(U)} = \{\text{families of morphisms } \{U_i \to U\} \text{ of } \mathscr{C} \\ \text{which are universal effective epimorphisms}\}.$$

Then every presheaf $\widetilde{W} = \mathrm{Hom}_{\mathscr{C}}(\cdot, W) \in \hat{\mathscr{C}}$ becomes a sheaf with respect to

$$\overline{\mathrm{Cov}(\mathscr{C})} = \{\overline{\mathrm{Cov}(U)} \mid U \in \mathrm{Ob}(\mathscr{C})\}.$$

Note that $(\mathscr{C}, \overline{\mathrm{Cov}(\mathscr{C})})$ becomes a site in the sense of Section 1.11, i.e., $\overline{\mathrm{Cov}(U)}$ satisfies (Site.1)–(Site.3). Then $(\mathscr{C}, \overline{\mathrm{Cov}(\mathscr{C})})$ is said to be a *canonical site*.

1.14 Sheaves of Abelian Groups

We have considered the category $\hat{\mathscr{C}} = \mathsf{Set}^{\mathscr{C}^\circ}$. In this section we will treat the case $\hat{\mathscr{T}} = \mathsf{Ab}^{\hat{\mathscr{T}}^\circ}$ where Ab is the category of abelian groups and $\mathscr{T}$ is the category associated to a topological space T. See Section 1.10 through Definition 5 and Examples 1.

An object $F \in \mathrm{Ob}(\hat{\mathscr{T}})$ is a contravariant functor from $\mathscr{T}$ to Ab. Therefore, for an object U (i.e., an open set) of $\mathscr{T}$, $F(U)$ is an abelian group and for a morphism $F \xrightarrow{\phi} G$ in $\hat{\mathscr{T}}$, $F(U) \xrightarrow{\phi_U} G(U)$ is a group homomorphism ϕ_U of abelian groups. Namely, a natural transformation ϕ (which will be called a morphism of presheaves) of presheaves F and G induces the group homomorphism ϕ_U over U from $F(U)$ to $G(U)$. Then define

$$(\ker \phi)(U) := \ker \phi_U = \{a_U \in F(U) \mid \phi_U(a_U) = 0_{G(U)}\}, \tag{14.1}$$

where $0_{G(U)}$ is a zero element of the abelian group $G(U)$. For $\iota : U \hookrightarrow V$ in $\mathscr{T}$ we have

$$\begin{array}{ccccc} (\ker \phi)(U) & = & \ker \phi_U \subset F(U) & \xrightarrow{\phi_U} & G(U) \\ & & \uparrow \; F\iota = \rho^V_U & & \uparrow \; G\iota = {}'\rho^V_U \\ (\ker \phi)(V) & = & \ker \phi_V \subset F(V) & \xrightarrow{\phi_V} & G(V) \end{array} \tag{14.2}$$

To show that $\ker \phi$ is a presheaf (a contravariant functor) we need the homomorphism $(\ker \phi)\iota : (\ker \phi)(V) \to (\ker \phi)(U)$ in (14.2). Let

$$a_V \in \ker \phi_V \subset F(V).$$

Then $\rho^V_U(a_V) \in F(U)$. Compute $\phi_U(\rho^V_U(a_V))$ by the commutativity of (14.2):

$$\phi_U(\rho^V_U(a_V)) = {}'\rho^V_U(\phi_V(a_V)) = {}'\rho^V_U(0_{G(V)}) = 0_{G(U)}. \tag{14.3}$$

Namely, $\rho_U^V(a_V) \in \ker\phi_U = (\ker\phi)(U)$. Define

$$((\ker\phi)\iota)(a_V) := \rho_U^V(a_V) \in (\ker\phi)(U). \tag{14.4}$$

Consequently for $\iota : U \hookrightarrow V$ in $\mathscr{T}$ we have in Ab

$$\begin{aligned} (\ker\phi)(V) &\xrightarrow{(\ker\phi)\iota} (\ker\phi)(U) \\ a_V &\longmapsto \rho_U^V(a_V). \end{aligned} \tag{14.5}$$

This assignment on an object and a morphism satisfies the presheaf axioms (PreSh1)–(PreSh2′) in Definition 4, i.e., $\ker\phi \in \mathsf{Ab}^{\mathscr{T}^\circ} = \hat{\mathscr{T}}$.

When F and G are sheaves, we will show that the presheaf $\ker\phi$ becomes a sheaf. Let $F \xrightarrow{\phi} G$ be a morphism of sheaves. For an open set U let $U = \cup_{i\in I}U_i$ be an arbitrary covering of U where $U, U_i \in \mathrm{Ob}(\mathscr{T})$. For $s_i \in (\ker\phi)(U_i)$, $i \in I$, assume $\rho^{U_i}_{U_i\cap U_j}(s_i) = \rho^{U_j}_{U_i\cap U_j}(s_j)$. Since s_i, s_j belong to $F(U_i)$ and $F(U_j)$ respectively, there exists a unique $s \in F(U)$ satisfying $\rho^U_{U_i}(s) = s_i$ for all $i \in I$. We need to show that this s belongs to $(\ker\phi)(U)$. Consider the following commutative diagram:

$$\begin{array}{ccccc} (\ker\phi)(U) = \ker\phi_U \subset & F(U) & \xrightarrow{\phi_U} & G(U) \\ & \downarrow{\scriptstyle \rho^U_{U_i}} & & \downarrow{\scriptstyle '\rho^U_{U_i}} \\ & F(U_i) & \xrightarrow{\phi_{U_i}} & G(U_i). \end{array} \tag{14.6}$$

For $s \in F(U)$ in (14.6) we have $\phi_{U_i}(\rho^U_{U_i}(s)) = \phi_{U_i}(s_i) = 0_{G(U_i)}$. In the the other direction in (14.6) we must have $'\rho^U_{U_i}(\phi_U(s)) = 0_{G(U_i)}$ by the commutativity. For $0_{G(U)} \in G(U)$ we also have $'\rho^U_{U_i}(0_{G(U)}) = 0_{G(U_i)}$. By the uniqueness in Definition 5 we have that $\phi_U(s) = 0_{G(U)}$, i.e.,

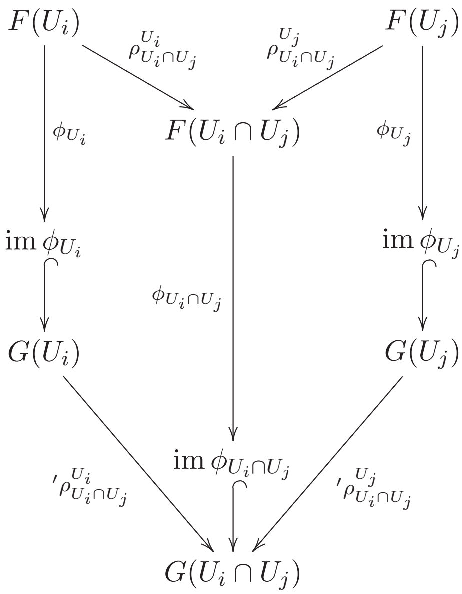
$$s \in \ker\phi_U = (\ker\phi)(U).$$

Consequently, the presheaf $\ker\phi$ is a sheaf.

Let $F \xrightarrow{\phi} G$ be a morphism of presheaves. Then as before, for $U \in \mathrm{Ob}(\mathscr{T})$ we have the group homomorphism $F(U) \xrightarrow{\phi_U} G(U)$ in Ab. Define

$$(\mathrm{im}\,\phi)(U) := \mathrm{im}\,\phi_U = \{\phi_U(s_U) \in G(U) \mid s_U \in F(U)\}. \tag{14.7}$$

Then $\mathrm{im}\,\phi : \mathscr{T}^\circ \rightsquigarrow \mathsf{Ab}$ is a presheaf. Even if F and G are sheaves, $\mathrm{im}\,\phi$ need not be a sheaf. In the following we will show why $\mathrm{im}\,\phi$ is not in general a sheaf. As before let $U = \cup_{i\in I}U_i$ be an open covering of U. Suppose that for $s'_i \in (\mathrm{im}\,\phi)(U_i) = \mathrm{im}\,\phi_{U_i}$, $i \in I$, $'\rho^{U_i}_{U_i\cap U_j}(s'_i) = \rho^{U_j}_{U_i\cap U_j}(s'_j)$ holds. Consider the following commutative diagram:



(14.8)

By regarding $s'_i \in (\operatorname{im}\phi)(U_i)$ and $s'_j \in (\operatorname{im}\phi)(U_j)$ as the sections of the sheaf G over U_i and U_j we find a unique $s' \in G(U)$ satisfying ${}'\rho^U_{U_i}(s') = s'_i \in \operatorname{im}\phi_{U_i}$ for all $i \in I$. The sheaf condition on $\operatorname{im}\phi$ is to claim $s' \in (\operatorname{im}\phi)(U) = \operatorname{im}\phi_U$. Namely, in order for $\operatorname{im}\phi$ to be a sheaf, $\phi_U : F(U) \to G(U)$ needs to be epimorphic for all U. As we will show in Chapter III, even if $\phi : F \to G$ is an epimorphism of sheaves, the induced homomorphism $\phi_U : F(U) \to G(U)$ of abelian groups need not be an epimorphism in Ab.

We define the presheaf $\operatorname{coker}\phi$ of a morphism of presheaves $\phi : F \to G$ by

$$(\operatorname{coker}\phi)(U) := \operatorname{coker}\phi_U = G(U)/\operatorname{im}\phi_U. \tag{14.9}$$

Even when $\phi : F \to G$ is a morphism of sheaves, $\operatorname{coker}\phi$ need not be a sheaf. We will demonstrate this situation as follows. As before we let $U = \cup_{i\in I} U_i$. Suppose that the class $\bar{s}'_{U_i} \in \operatorname{coker}\phi_{U_i}$ of $s'_{U_i} \in G(U_i)$ is $\overline{0_{U_i}}$. Namely,

$$\bar{s}'_{U_i} = s'_{U_i} + \operatorname{im}\phi_{U_i} = \overline{0_{U_i}}.$$

Then we have $s'_{U_i} \in \operatorname{im}\phi_{U_i}$. Suppose that the induced homomorphisms of the restriction maps satisfy

$$\overline{{}'\rho^{U_i}_{U_i\cap U_j}}(\overline{0_{U_i}}) = \overline{{}'\rho^{U_j}_{U_i\cap U_j}}(\overline{0_{U_j}}) \tag{14.10}$$

in $\operatorname{coker}\phi_{U_i\cap U_j} = G(U_i \cap U_j)/\operatorname{im}\phi_{U_i\cap U_j}$. Namely, each $s'_{U_i} \in \operatorname{im}\phi_{U_i}$ satisfies ${}'\rho^{U_i}_{U_i\cap U_j}(s'_{U_i}) = {}'\rho^{U_j}_{U_i\cap U_j}(s'_{U_j})$ as in the above paragraph. Since $\operatorname{im}\phi$ need not be a sheaf, $\operatorname{coker}\phi$ also need not be a sheaf.

1.15 The Sheafification Functor

For the inclusion functor from the category $\tilde{\mathscr{T}}$ of sheaves to the category $\hat{\mathscr{T}} = \mathsf{Ab}^{\mathscr{T}^\circ}$, i.e.,

$$\iota : \tilde{\mathscr{T}} \rightsquigarrow \hat{\mathscr{T}} \tag{15.1}$$

we will construct a functor

$$\mathsf{sh} : \hat{\mathscr{T}} \rightsquigarrow \tilde{\mathscr{T}} \tag{15.2}$$

so that the inclusion functor ι may be the (right) adjoint to the functor sh as in Section 1.7. That is, in the diagram

$$\begin{array}{ccc} \hat{\mathscr{T}} & \overset{\mathsf{sh}}{\rightsquigarrow} & \tilde{\mathscr{T}} \\ \widetilde{\iota G} \searrow & & \swarrow \widetilde{G}=\mathrm{Hom}_{\tilde{\mathscr{T}}}(\cdot, G) \\ & \mathsf{Ab} & \end{array} \tag{15.3}$$

we have in $\hat{\hat{\mathscr{T}}} = \mathsf{Ab}^{\hat{\mathscr{T}}^\circ}$

$$\widetilde{\iota G} \approx \widetilde{G} \circ \mathsf{sh}, \tag{15.4}$$

where (15.4) means that for any presheaf F

$$\widetilde{\iota G}(F) = \mathrm{Hom}_{\hat{\mathscr{T}}}(F, \iota G) \approx (\widetilde{G} \circ \mathsf{sh})(F) = \mathrm{Hom}_{\tilde{\mathscr{T}}}(\mathsf{sh}F, G). \tag{15.5}$$

Compare (15.4) with (7.5) and (15.3) with (7.8).

Let F be a presheaf, and let U be an open set. Then define $(\mathsf{sh}F)(U)$ as the set of all mappings s from U to the direct product $\prod_{x \in U} F_x$ of stalks satisfying $s(x) \in F_x$ and the following gluing condition (Glue) for condition (Sheaf) in Definition 5.

(Glue) For $x \in U$, there is an open set W contained in U and there exists a section $t \in F(W)$ so that for every point $x' \in W$, $s(x')$ is the germ of t at x', i.e., we have $t_{x'} = s(x') \in F_{x'}$.

We will show that $\mathsf{sh}F$ is indeed a sheaf. Let $U = \cup_{i \in I} U_i$ be any covering of U. For $s_i \in \mathsf{sh}F(U_i)$, $i \in I$, suppose $\bar{\rho}^{U_i}_{U_i \cap U_j}(s_i) = \bar{\rho}^{U_j}_{U_i \cap U_j}(s_j)$ holds where $\bar{\rho}$ is induced by $\bar{\rho}^U_V = \mathsf{sh}F(\iota) : \mathsf{sh}F(U) \to \mathsf{sh}F(V)$ for $\iota : V \hookrightarrow U$. Then by (Glue) there exist $W \subset U_i \cap U_j$ and $t \in F(W)$ satisfying

$$\bar{\rho}^{U_i}_W(s_i)(x') = t_{x'} = \bar{\rho}^{U_j}_W(s_j)(x')$$

for all $x' \in W$. This t can be used to glue $s_i \in \mathsf{sh}F(U_i)$ and $s_j \in \mathsf{sh}F(U_j)$ to get $s_{i \cup j} \in \mathsf{sh}F(U_i \cup U_j)$. Consequently, we obtain an $s \in \mathsf{sh}F(U)$ to satisfy $\bar{\rho}^U_{U_i}(s) = s_i$.

1.15.1 Universality for sh$F$

The isomorphism in (15.5), i.e., $\mathrm{Hom}_{\hat{\mathscr{T}}}(F, \iota G) \approx \mathrm{Hom}_{\tilde{\mathscr{T}}}(\mathsf{sh}F, G)$, implies that for $\phi \in \mathrm{Hom}_{\hat{\mathscr{T}}}(F, \iota G)$ there exists a unique morphism $\psi : \mathsf{sh}F \to G$ of sheaves. Namely, in the category $\hat{\mathscr{T}}$ of presheaves we have

$$\begin{array}{ccc} F & \xrightarrow{\;\theta\;} & \mathsf{sh}F \\ & {\scriptstyle\phi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & \iota G & \end{array} \tag{15.6}$$

where the morphism θ (which is a natural transformation of objects in $\hat{\mathscr{T}}$) is defined as follows. For an open set U, we have $\theta_U : F(U) \to \mathsf{sh}F(U)$ in Ab defined by

$$\begin{aligned} \theta_U(s) : U &\longrightarrow \prod_{x \in U} F_x, \qquad s \in F(U) \\ x &\longmapsto s_x. \end{aligned}$$

Such an object $\mathsf{sh}F$ satisfying the above universal mapping property, which is uniquely determined, can be used as a definition of a *sheafification* of a presheaf. That is, the sheafification $\mathsf{sh}F$ of a presheaf F is a sheaf $\mathsf{sh}F$ satisfying (15.5) for any sheaf G i.e., (15.6).

Remark 14. The inclusion functor ι which regards a sheaf just as a presheaf is a left exact functor from $\tilde{\mathscr{T}}$ to $\hat{\mathscr{T}}$ in the following sense. For an exact sequence

$$0 \longrightarrow F' \xrightarrow{\;\phi\;} F \xrightarrow{\;\psi\;} F'' \longrightarrow 0$$

as sheaves, we have only the following exactness

$$0 \longrightarrow \iota F' \xrightarrow{\;\iota\phi\;} \iota F \xrightarrow{\;\iota\psi\;} \iota F''$$

as presheaves. This means that for an open set U the sequence of presheaves

$$0 \longrightarrow F'(U) \xrightarrow{\;\phi_U\;} F(U) \xrightarrow{\;\psi_U\;} F''(U)$$

is exact in Ab. This topic will be treated in Chapter III.

Chapter 2

DERIVED FUNCTORS

2.1 Complexes

Let $\mathscr{A}$ be an abelian category. We will define the category $\mathsf{Co}(\mathscr{A})$ of (cochain) complexes as follows. An object in $\mathsf{Co}(\mathscr{A})$ is a sequence of objects and morphisms

$$\cdots \longrightarrow A^{j-1} \xrightarrow{\mathrm{d}^{j-1}} A^j \xrightarrow{\mathrm{d}^j} A^{j+1} \xrightarrow{\mathrm{d}^{j+1}} \cdots \tag{1.1}$$

such that $A^j \in \mathrm{Ob}(\mathscr{A})$ and $\mathrm{d}^j \in \mathrm{Hom}_{\mathscr{A}}(A^j, A^{j+1})$ satisfying $\mathrm{d}^j \circ \mathrm{d}^{j-1} = 0$ for all $j \in \mathbb{Z}$, the set of integers. We often write the object in (1.1) as $A^\bullet$. A morphism between objects $A^\bullet$ and $B^\bullet$ in $\mathsf{Co}(\mathscr{A})$ is defined as a collection of morphisms $f^j : A^j \to B^j$ in $\mathscr{A}$ for $j \in \mathbb{Z}$ so that in

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & A^{j-1} & \xrightarrow{\mathrm{d}^{j-1}} & A^j & \xrightarrow{\mathrm{d}^j} & A^{j+1} & \xrightarrow{\mathrm{d}^{j+1}} & \cdots \\
 & & \downarrow{\scriptstyle f^{j-1}} & & \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{j+1}} & & \\
\cdots & \longrightarrow & B^{j-1} & \xrightarrow{'\mathrm{d}^{j-1}} & B^j & \xrightarrow{'\mathrm{d}^j} & B^{j+1} & \xrightarrow{'\mathrm{d}^{j+1}} & \cdots
\end{array} \tag{1.2}$$

we have that $f^j \circ \mathrm{d}^{j-1} = {}'\mathrm{d}^{j-1} \circ f^{j-1}$ for all $j \in \mathbb{Z}$. We often write (1.2) as $f^\bullet : A^\bullet \to B^\bullet$.

2.2 Cohomology

Let $A^\bullet \in \mathrm{Ob}(\mathsf{Co}(\mathscr{A}))$. Since $\mathrm{d}^j \circ \mathrm{d}^{j-1} = 0$, $\mathrm{im}\,\mathrm{d}^{j-1} \subset \ker \mathrm{d}^j$ holds. Therefore, we can consider the quotient object

$$\ker \mathrm{d}^j \big/ \mathrm{im}\,\mathrm{d}^{j-1}. \tag{2.1}$$

Since $\ker \mathrm{d}^j \subset A^j$, the object (2.1) is a subquotient object of A^j. We define the *j-th cohomology* $\mathrm{H}^j(A^\bullet)$ as

$$\mathrm{H}^j(A^\bullet) := \ker \mathrm{d}^j \big/ \operatorname{im} \mathrm{d}^{j-1}.$$

Then for each j

$$\begin{aligned} \mathrm{H}^j : \mathsf{Co}(\mathscr{A}) &\rightsquigarrow \mathscr{A} \\ A^\bullet &\longmapsto \mathrm{H}^j(\mathscr{A}^\bullet) \end{aligned} \tag{2.2}$$

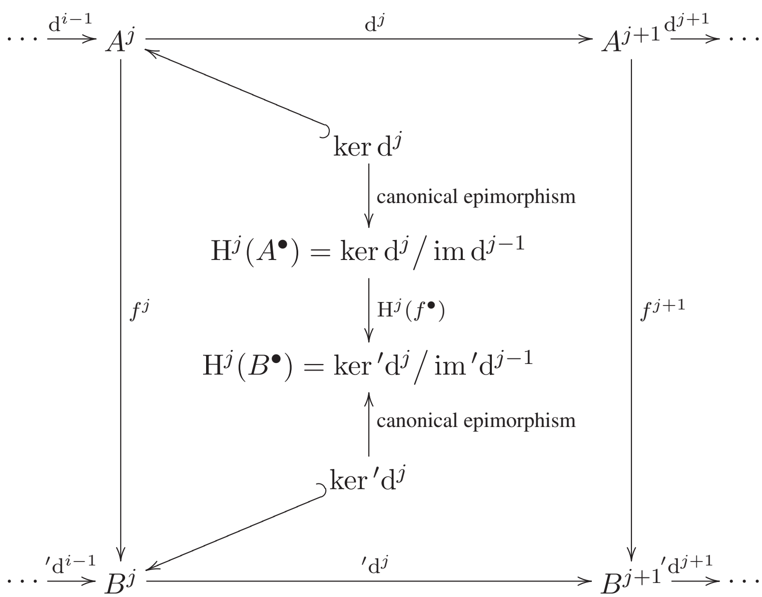
is a functor. Let $f^\bullet : A^\bullet \to B^\bullet$ be a morphism in $\mathsf{Co}(\mathscr{A})$. Then the induced morphism $\mathrm{H}^j(f^\bullet) : \mathrm{H}^j(A^\bullet) \to \mathrm{H}^j(B^\bullet)$ in $\mathscr{A}$ is given as follows. For

$$\bar{x} \in \mathrm{H}^j(A^\bullet) = \ker \mathrm{d}^j \big/ \operatorname{im} \mathrm{d}^{j-1},$$

where $\bar{x}$ is the class of $x \in \ker \mathrm{d}^j$, we have that $\mathrm{H}^j(f^\bullet)\bar{x} = \overline{f^j(x)}$, where $\overline{f^j(x)}$ is the class of $f^j(x)$ in $\ker {}'\mathrm{d}^j \big/ \operatorname{im} {}'\mathrm{d}^{j-1}$. Notice that since the commutativity of the diagram (1.2), i.e.,

$${}'\mathrm{d}^j(f^j(x)) = f^{j+1}(\mathrm{d}^j(x)),$$

for $x \in \ker \mathrm{d}^j$, we have ${}'\mathrm{d}^j(f^j(x)) = 0$ in B^{j+1}. See the following diagram for the above computation.



(2.3)

Namely, $\mathrm{H}^j : \mathsf{Co}(\mathscr{A}) \rightsquigarrow \mathscr{A}$ is a covariant functor which is said to be a cohomological functor.

Notation 12. Let $\mathsf{Co}^+(\mathscr{A})$ be the category whose objects consist of complexes bounded from below, i.e., $A^\bullet = (A^j)_{j\geq 0}$:

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A^0 \xrightarrow{\mathrm{d}^0} A^1 \xrightarrow{\mathrm{d}^1} \cdots, \quad A^j \in \mathrm{Ob}(\mathscr{A}),$$

is an object of $\mathsf{Co}^+(\mathscr{A})$.

2.3 Homotopy

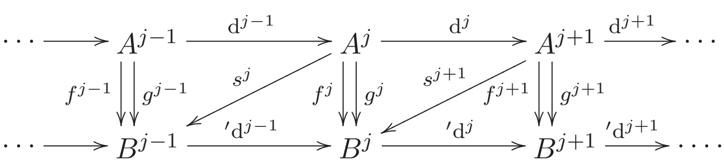
Let $A^\bullet$ and $B^\bullet$ be objects in $\mathsf{Co}(\mathscr{A})$ and let $f^\bullet$ and $g^\bullet$ be morphisms from $A^\bullet$ to $B^\bullet$. The functor H^j induces the morphisms $\mathrm{H}(f^\bullet)$ and $\mathrm{H}^j(g^\bullet)$ from $\mathrm{H}^j(A^\bullet)$ to $\mathrm{H}^j(B^\bullet)$ in $\mathscr{A}$. We ask when we get $\mathrm{H}^j(f^\bullet) = \mathrm{H}^j(g^\bullet)$. Using the notation in Section 2.2, for an arbitrary $\bar{x} \in \mathrm{H}^j(A^\bullet)$, the equality $\mathrm{H}^j(f^\bullet) = \mathrm{H}^j(g^\bullet)$ can be phrased as: for $\bar{x} \in \mathrm{H}^j(A^\bullet) = \ker \mathrm{d}^j \big/ \operatorname{im} \mathrm{d}^{j-1}$,

$$\mathrm{H}^j(f^\bullet)\bar{x} = \overline{f^j(x)} = \overline{g^j(x)} = \mathrm{H}^j(g^\bullet)\bar{x} \tag{3.1}$$

in $\mathrm{H}^j(B^\bullet)$. Namely, (3.1) means that the cohomology classes of $f^j(x)$ and $g^j(x)$ are the same, i.e., $f^j(x) - g^j(x) \in \operatorname{im}{}'\mathrm{d}^{j-1}$. Let $s^j : A^j \to B^{j-1}$ be a morphism in $\mathscr{A}$. Then ${}'\mathrm{d}^{j-1} \circ s^j + s^{j+1} \circ \mathrm{d}^j$ is a morphism from A^j to B^j, $j \in \mathbb{Z}$. We then assert: if

$$f^j - g^j = {}'\mathrm{d}^{j-1} \circ s^j + s^{j+1} \circ \mathrm{d}^j, \tag{3.2}$$

then $\mathrm{H}^j(f^\bullet) = \mathrm{H}^j(g^\bullet)$ holds. See the diagram below.



(3.3)

For $x \in \ker \mathrm{d}^j$, let us compute (3.2) as follows:

$$(f^j - g^j)(x) = f^j(x) - g^j(x) = {}'\mathrm{d}^{j-1}(s^j(x)) + s^{j+1}(\mathrm{d}^j(x)) = {}'\mathrm{d}^{j-1}(s^j(x)).$$

Since $s^j(x) \in B^{j-1}$ we have $f^j(x) - g^j(x) \in \operatorname{im}{}'\mathrm{d}^{j-1}$. That is, (3.2), implies $\mathrm{H}^j(f^\bullet) = \mathrm{H}^j(g^\bullet)$. Morphisms $f^\bullet, g^\bullet \in \mathrm{Hom}_{\mathsf{Co}(\mathscr{A})}(A^\bullet, B^\bullet)$ are said to be *homotopic* if we have $\mathrm{H}^j(f^\bullet) = \mathrm{H}^j(g^\bullet)$ as morphisms from $\mathrm{H}^j(A^\bullet)$ to $\mathrm{H}^j(B^\bullet)$. When $f^\bullet$ is homotopic to $g^\bullet$, we write $f^\bullet \sim g^\bullet$. Notice that $\sim$ is an equivalence relation in the set $\mathrm{Hom}_{\mathsf{Co}(\mathscr{A})}(A^\bullet, B^\bullet)$. We define

$$\mathsf{K}(\mathscr{A}) := \mathsf{Co}(\mathscr{A}) \big/ \sim . \tag{3.4}$$

That is, the objects of $\mathsf{K}(\mathscr{A})$ are precisely the objects of $\mathsf{Co}(\mathscr{A})$ and morphisms are the homotopy equivalence classes of morphisms as we defined above. Then the functor $\mathrm{H}^j : \mathsf{Co}(\mathscr{A}) \rightsquigarrow \mathscr{A}$ in Section 2.1 can be extended to

$${}'\mathrm{H}^j : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathscr{A} \tag{3.5}$$

defined by $'\mathrm{H}^j(\overline{f^\bullet}) = \mathrm{H}^j(f^\bullet)$. We will use the same H^j for both functors from $\mathsf{Co}(\mathscr{A})$ and from $\mathsf{K}(\mathscr{A})$ to $\mathscr{A}$.

2.4 Exactness

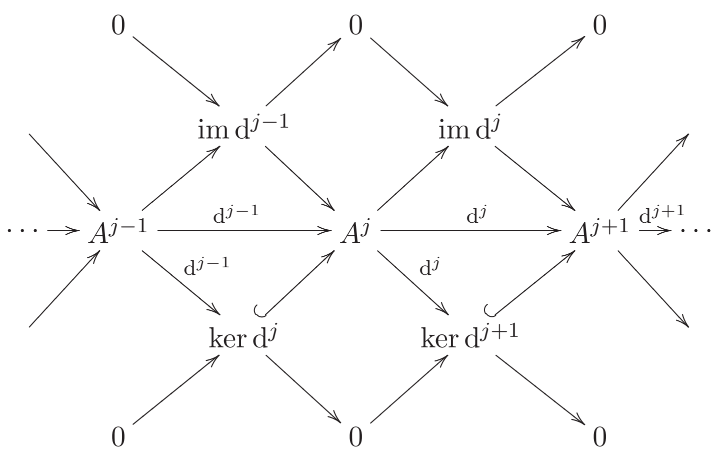
Let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories. Recall the following from Section 1.4: a functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ is said to be additive if the induced homomorphism $\bar{F}$ by F is a group homomorphism from $\mathrm{Hom}_{\mathscr{A}}(A, B)$ to $\mathrm{Hom}_{\mathscr{B}}(FA, FB)$ for $A, B \in \mathrm{Ob}(\mathscr{A})$. Our interest is to measure the loss of exactness as F takes an object of $\mathscr{A}$ into an object of $\mathscr{B}$. Namely, for an exact sequence

$$\cdots \longrightarrow A^{j-1} \xrightarrow{\mathrm{d}^{j-1}} A^j \xrightarrow{\mathrm{d}^j} A^{j+1} \xrightarrow{\mathrm{d}^{j+1}} \cdots$$

in $\mathscr{A}$, we measure the loss of exactness of

$$\cdots \longrightarrow FA^{j-1} \xrightarrow{F\mathrm{d}^{j-1}} FA^j \xrightarrow{F\mathrm{d}^j} FA^{j+1} \xrightarrow{F\mathrm{d}^{j+1}} \cdots$$

in $\mathscr{B}$ by calculating the cohomology $\mathrm{H}^j(FA^\bullet) = \ker F\mathrm{d}^j \big/ \operatorname{im} F\mathrm{d}^{j-1}$, a subquotient object of FA^j. For a complex $A^\bullet \in \mathrm{Ob}(\mathsf{Co}(\mathscr{A}))$ (since F is a functor) we have $0 = F(\mathrm{d}^j \circ \mathrm{d}^{j-1}) = F\mathrm{d}^j \circ F\mathrm{d}^{j-1}$. That is $FA^\bullet$ is a complex, i.e., $FA^\bullet \in \mathrm{Ob}(\mathsf{Co}(\mathscr{B}))$. Next for any complex $A^\bullet$ of objects and morphisms of $\mathscr{A}$, we can decompose the complex $A^\bullet$ as follows:



(4.1)

where all the diagonal short sequences are exact. Therefore, it is sufficient to consider the effect of F on a short exact sequence

$$0 \longrightarrow A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \longrightarrow 0 \qquad (4.2)$$

in $\mathscr{A}$ to measure the loss of the exactness of

$$0 \longrightarrow FA' \xrightarrow{F\phi} FA \xrightarrow{F\psi} FA'' \longrightarrow 0 \qquad (4.3)$$

in $\mathscr{B}$. A functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ is said to be *exact* when (4.3) is exact in $\mathscr{B}$ at FA', FA and FA''. Namely, $F\phi$ is a monomorphism, $\ker F\psi = \operatorname{im} F\phi$ and $F\psi$ is an epimorphism in $\mathscr{B}$. When only

$$0 \longrightarrow FA' \xrightarrow{F\phi} FA \xrightarrow{F\psi} FA''$$

is exact, i.e., $F\psi$ need not be an epimorphism, F is said to be a *left exact* functor. Similarly, when

$$FA' \xrightarrow{F\phi} FA \xrightarrow{F\psi} FA'' \longrightarrow 0$$

is exact in $\mathscr{B}$, F is said to be a *right exact* functor. When only at FA, the exactness is preserved (i.e., if we only have $\operatorname{im} F\phi = \ker F\psi$ in (4.3)), F is said to be *half-exact*.

2.5 Injective Objects

[Injective Objects] Let $\mathscr{A}$ be an abelian category. Then for objects A and B in $\mathscr{A}$, $\operatorname{Hom}_{\mathscr{A}}(B, A)$ is an abelian group (i.e., condition (A.1) of Section 1.6). Then the contravariant functor $\operatorname{Hom}_{\mathscr{A}}(\cdot, A)$ is a left exact functor from $\mathscr{A}$ to Ab. That is, for an arbitrary short exact sequence in $\mathscr{A}$

$$0 \longrightarrow C' \xrightarrow{\phi} C \xrightarrow{\psi} C'' \longrightarrow 0 \tag{5.1}$$

we have the exact sequence in Ab

$$0 \longrightarrow \operatorname{Hom}_{\mathscr{A}}(C'', A) \xrightarrow{\psi^*} \operatorname{Hom}_{\mathscr{A}}(C, A) \xrightarrow{\phi^*} \operatorname{Hom}_{\mathscr{A}}(C', A)\ , \tag{5.2}$$

where, for instance, $\phi^* := \operatorname{Hom}_{\mathscr{A}}(\phi, A)$. One may like to interpret the exactness of (5.2) through the following commutative diagram:

$$\begin{tikzcd} & & A & & \\ 0 \arrow[r] & C' \arrow[r, "\phi"] \arrow[ur] & C \arrow[r, "\psi"] \arrow[u] & C'' \arrow[r] \arrow[ul] & 0. \end{tikzcd} \tag{5.3}$$

An *injective object* I in $\mathscr{A}$ is an object to guarantee the exactness of the functor $\operatorname{Hom}_{\mathscr{A}}(\cdot, I) : \mathscr{A} \rightsquigarrow \mathsf{Ab}$. That is, in the diagram

$$\begin{tikzcd} & & I \\ 0 \arrow[r] & C' \arrow[r, "\phi"] \arrow[ur, "f'"] & C \arrow[u, dotted, "f"'] \end{tikzcd} \tag{5.4}$$

any morphism $f' : C' \to I$ can be lifted to $f : C \to I$ satisfying $f' = f \circ \phi$. Namely, the induced morphism ϕ^* in (5.2) from ϕ becomes an epimorphism: also

$$\mathrm{Hom}_A(C, I) \xrightarrow{\phi^*} \mathrm{Hom}_{\mathscr{A}}(C', I) \longrightarrow 0$$

is exact. That is, an object I is said to be an injective object if

$$\mathrm{Hom}_{\mathscr{A}}(\cdot, I) : \mathscr{A} \rightsquigarrow \mathsf{Ab}$$

becomes an exact functor.

Dually, an object P is said to be a *projective object* of $\mathscr{A}$ if the covariant functor $\mathrm{Hom}_{\mathscr{A}}(P, \cdot) : \mathscr{A} \rightsquigarrow \mathsf{Ab}$ becomes exact. Namely, for a short exact sequence in $\mathscr{A}$ as in (5.1) the induced sequence in Ab:

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{A}}(P, C') \xrightarrow{\phi_*} \mathrm{Hom}_{\mathscr{A}}(P, C) \xrightarrow{\psi_*} \mathrm{Hom}_{\mathscr{A}}(P, C'') \longrightarrow 0\ , \qquad (5.5)$$

is exact, where, for instance, $\phi_* := \mathrm{Hom}_{\mathscr{A}}(P, \phi)$.

Note 13. Let $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ be adjoint to $G : \mathscr{C}' \rightsquigarrow \mathscr{C}$. Suppose that F takes monomorphisms in $\mathscr{C}$ to monomorphisms in $\mathscr{C}'$ (e.g., F is an exact functor). Then G takes injective objects of $\mathscr{C}'$ to injective objects of $\mathscr{C}$. We will prove this assertion as follows. Let I' be an injective object of $\mathscr{C}'$ and let

$$0 \longrightarrow C' \xrightarrow{\phi} C \xrightarrow{\psi} C'' \longrightarrow 0$$

be an arbitrary short exact sequence in $\mathscr{C}$. By the assumption, we have the exact sequence

$$0 \longrightarrow FC' \xrightarrow{F\phi} FC \xrightarrow{F\psi} FC''$$

in $\mathscr{C}'$. Since the contravariant functor $\mathrm{Hom}_{\mathscr{C}'}(\cdot, I')$ is an exact functor, we get the exact sequence

$$\mathrm{Hom}_{\mathscr{C}'}(FC'', I') \xrightarrow{(F\psi)^*} \mathrm{Hom}_{\mathscr{C}'}(FC, I') \xrightarrow{(F\phi)^*} \mathrm{Hom}_{\mathscr{C}'}(FC', I') \longrightarrow 0$$

in Ab. Consider,

$$\begin{array}{ccccccc} & \mathrm{Hom}_{\mathscr{C}'}(FC'', I') & \to & \mathrm{Hom}_{\mathscr{C}'}(FC', I') & \to & \mathrm{Hom}_{\mathscr{C}'}(FC', I') & \to 0 \\ & \approx\uparrow & & \approx\uparrow & & \approx\uparrow & \\ 0 \to & \mathrm{Hom}_{\mathscr{C}}(C'', GI') & \xrightarrow{\psi^*} & \mathrm{Hom}_{\mathscr{C}}(C, GI') & \xrightarrow{\phi^*} & \mathrm{Hom}_{\mathscr{C}}(C', GI') & \to 0 \end{array} \qquad (5.6)$$

where the vertical homomorphisms are isomorphisms (i.e., (7.7) in Chapter I). Therefore, ϕ^* in the induced sequence in (5.6) becomes an epimorphism. In general, since $\mathrm{Hom}_{\mathscr{C}}(\cdot, GI')$ is left exact, we conclude that $\mathrm{Hom}_{\mathscr{C}}(\cdot, GI')$ is an exact functor. Namely, GI' is an injective object of $\mathscr{C}$ for an injective object I' in $\mathscr{C}'$.

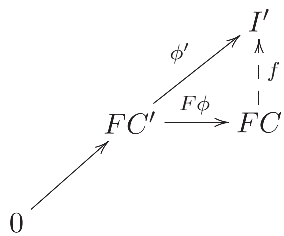
Note 14. An abelian category $\mathscr{C}$ is said to have *enough injectives* if for an arbitrary object A of $\mathscr{C}$ there exists a monomorphism from A to an injective object I of $\mathscr{C}$. Namely A is a subobject of I. A category is said to have *enough projectives* if an arbitrary A is a quotient object of a projective object P of $\mathscr{C}$. Namely, there exists an epimorphism from P to A. An injective object of $\mathscr{C}$ is projective object in the dual category $\mathscr{C}^\circ$. Consequently, $\mathscr{C}$ has enough injectives if and only if $\mathscr{C}^\circ$ has enough projectives. We can prove the following assertion: as in Note 13, let $F : \mathscr{C} \rightsquigarrow \mathscr{C}'$ be adjoint to $G : \mathscr{C}' \rightsquigarrow \mathscr{C}$ and assume that $\mathscr{C}'$ has enough injectives. We have: if G takes injective objects of $\mathscr{C}'$ into injective objects of $\mathscr{C}$ then F takes monomorphisms in $\mathscr{C}$ to monomorphisms in $\mathscr{C}'$. Let $C' \xrightarrow{\phi} C$ be a monomorphism in $\mathscr{C}$. We shall prove that $F\phi : FC' \to FC'$ is a monomorphism in $\mathscr{C}'$. We have a monomorphism $FC' \xrightarrow{\phi'} I'$ in $\mathscr{C}'$, where I' is an injective object of $\mathscr{C}'$. Then GI' is injective in $\mathscr{C}$. Since $\mathrm{Hom}_{\mathscr{C}}(\cdot, GI')$ is an exact functor. For $0 \to C' \xrightarrow{\phi} C$, the induced

$$\mathrm{Hom}_{\mathscr{C}}(C, GI') \xrightarrow{\phi^*} \mathrm{Hom}_{\mathscr{C}}(C', GI') \longrightarrow 0$$

is an epimorphism. Since F and G are mutually adjoint, this epimorphism induces the epimorphism

$$\mathrm{Hom}_{\mathscr{C}'}(FC, I') \xrightarrow{(F\phi)^*} \mathrm{Hom}_{\mathscr{C}'}(FC', I') \longrightarrow 0.$$

That is, in particular, for $\phi' \in \mathrm{Hom}_{\mathscr{C}'}(FC', I')$, there is $f \in \mathrm{Hom}_{\mathscr{C}'}(FC, I')$ satisfying $(F\phi)^* f = \phi'$. Namely, $f \circ F\phi = \phi'$ in the following diagram



(5.7)

implying that $F\phi : FC' \to FC$ is a monomorphism.

Remark 15. We summarize Note 13 and Note 14 as follows. For the adjoint pair F and G, when $\mathscr{C}'$ has enough injectives, we have: F preserves monomorphisms if and only if G preserves injectives.

2.6 Resolutions

Let $\mathscr{A}$ be an abelian category and $A^\bullet$ and $B^\bullet$ be complexes, i.e., objects in $\mathsf{Co}(\mathscr{A})$, and let $f^\bullet$ be a morphism from $A^\bullet$ to $B^\bullet$. As was shown in Section

2.2 the morphism $\mathrm{H}^j(f^\bullet) : \mathrm{H}^j(A^\bullet) \to \mathrm{H}^j(B^\bullet)$ is induced. When $\mathrm{H}^j(f^\bullet)$ is an isomorphism, $j \in \mathbb{Z}$, $f^\bullet : A^\bullet \to B^\bullet$ is said to be a *quasi-isomorphism*. For a single object $A \in \mathrm{Ob}(\mathscr{A})$, we regard A as

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow A \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

i.e., an object of $\mathsf{Co}(\mathscr{A})$. When a complex $I^\bullet$ consisting of injective objects I^j, $j \geq 0$, of $\mathscr{A}$ is quasi-isomorphic to the complex A, $I^\bullet$ is said to be an *injective resolution* of A. That is, the morphism $(\ldots, 0, \epsilon, 0, \ldots)$ of $\mathsf{Co}(\mathscr{A})$ in the diagram

$$\begin{array}{ccccccccc} \cdots \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow \cdots \\ & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle \epsilon} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & \\ \cdots \longrightarrow & 0 & \longrightarrow & I^0 & \xrightarrow{\mathrm{d}^0} & I^1 & \xrightarrow{\mathrm{d}^1} & I^2 & \xrightarrow{\mathrm{d}^2} \cdots \end{array} \tag{6.1}$$

induces isomorphisms $\mathrm{H}^j(A^\bullet) \xrightarrow{\approx} \mathrm{H}^j(I^\bullet)$, $j \in \mathbb{Z}$. Namely, $I^\bullet$ is exact at each I^j, i.e., $\mathrm{H}^j(I^\bullet) = 0$, $j \neq 0$ and for $j = 0$ the induced morphism

$$\mathrm{H}^0(A) = A \xrightarrow{\epsilon} \mathrm{H}^0(I^\bullet) = \ker \mathrm{d}^0$$

is an isomorphism. Consequently, we have the isomorphism $A \approx \mathrm{im}\, \epsilon = \ker \mathrm{d}^0$. We often write an injective resolution of A as

$$0 \longrightarrow A \xrightarrow{\epsilon} I^0 \xrightarrow{\mathrm{d}^0} I^1 \xrightarrow{\mathrm{d}^1} \cdots \tag{6.2}$$

Namely, one may say that $I^\bullet \in \mathsf{Co}(\mathscr{A})$ is an injective resolution of an object A of $\mathscr{A}$ when (6.2) is an exact sequence in $\mathscr{C}$, and each I^j is an injective object. Notice that every object has an injective resolution in a category with enough injectives in the sense of Note 13.

2.7 Derived Functors

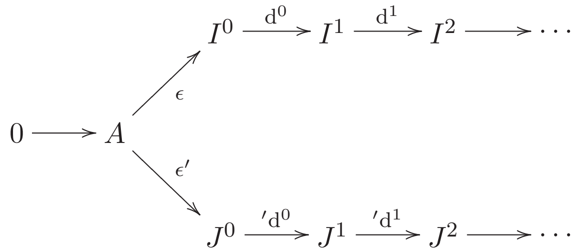
Let $\mathscr{A}$ be an abelian category and let F be an additive left exact functor from $\mathscr{A}$ to another abelian category $\mathscr{B}$. Assume that $\mathscr{A}$ has enough injectives. For an injective resolution $I^\bullet$ of an object A in $\mathscr{A}$, $FI^\bullet$ is a complex, i.e., $FI^\bullet \in \mathsf{Co}(\mathscr{B})$. We define the *$j$-th right derived functor* of F at A as

$$\mathrm{R}^j FA := \mathrm{H}^j(FI^\bullet) = \ker F\mathrm{d}^j \big/ \mathrm{im}\, F\mathrm{d}^{j-1} \tag{7.1}$$

where

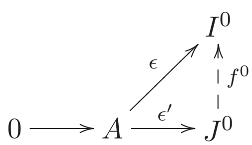
$$FI^\bullet : \cdots \longrightarrow 0 \longrightarrow FI^0 \xrightarrow{F\mathrm{d}^0} FI^1 \xrightarrow{F\mathrm{d}^1} \cdots.$$

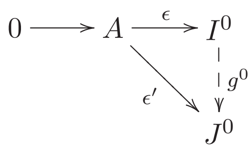
Let $I^\bullet$ and $J^\bullet$ be two injective resolutions of an object A in $\mathscr{A}$. To justify the notation $\mathrm{R}^j FA$ in (7.1) of the right derived functor of F at A, we will prove that $\mathrm{H}^j(FI^\bullet)$ and $\mathrm{H}^j(FJ^\bullet)$ are isomorphic for $j \geq 0$. Let us consider



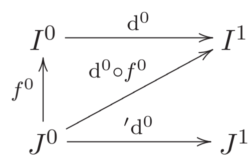
(7.2)

For the monomorphism ϵ' in (7.2), the injectiveness of I^0 implies that there exists a morphism f^0 from J^0 to I^0. Namely, we have the following commutative diagrams

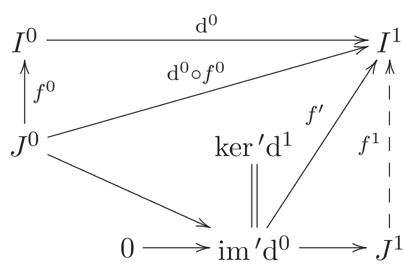




where the second diagram is obtained by the injectiveness of J^0. Then $\mathrm{d}^0 \circ f^0$ is a morphism from J^0 to I^1. From the diagram



we get

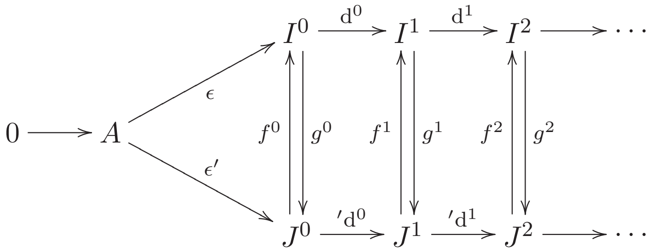


(7.3)

In order to show the existence of a morphism $f^1 : J^1 \to I^1$ in (7.3), first we will define a morphism $f' : \mathrm{im}\,'\mathrm{d}^0 \to I^1$ as follows. By using an Exact Embedding Theorem of an abelian category into the category of abelian groups, we define f' for an element $'\mathrm{d}^0(y^0) \in \mathrm{im}\,'\mathrm{d}^0$ where $y^0 \in J^0$, by

$$f'('\mathrm{d}^0(y^0)) := (\mathrm{d}^0 \circ f^0)(y^0).$$

Since I^1 is an injective object we get a morphism $f^1 : J^1 \to I^1$ as shown in (7.3). By exchanging the role of I^1 and J^1 we get similarly a morphism $g^1 : I^1 \to J^1$. Thus we obtain



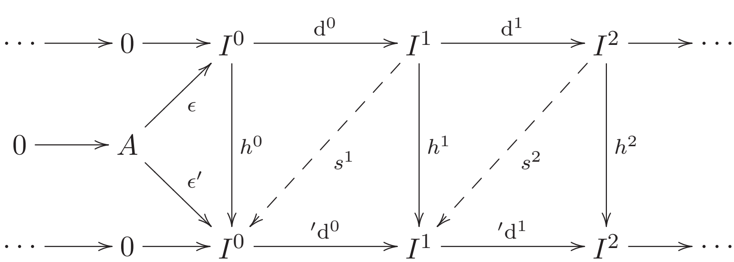
Then we have

$$FI^\bullet \underset{Ff^\bullet}{\overset{Fg^\bullet}{\rightleftarrows}} FJ^\bullet$$

in $\mathsf{Co}(\mathscr{B})$. We would like to show that the induced morphisms on cohomologies are isomorphisms. That is, we will prove the following:

$$\begin{aligned} \mathrm{H}^j(Ff^\bullet) \circ \mathrm{H}^j(Fg^\bullet) &= \mathrm{H}^j(Ff^\bullet \circ Fg^\bullet) = \\ &= \mathrm{H}^j(F(f^\bullet \circ g^\bullet)) = 1_{\mathrm{H}^j(FI^\bullet)} \\ \mathrm{H}^j(Fg^\bullet) \circ \mathrm{H}^j(Ff^\bullet) &= \mathrm{H}^j(Fg^\bullet \circ Ff^\bullet) = \\ &= \mathrm{H}^j(F(g^\bullet \circ f^\bullet)) = 1_{\mathrm{H}^j(FJ^\bullet)} \end{aligned} \tag{7.4}$$

For $f^\bullet \circ g^\bullet : I^\bullet \to I^\bullet$ and $1_{I^\bullet} : I^\bullet \to I^\bullet$, if $F(f^\bullet \circ g^\bullet)$ and $F(1_{I^\bullet})$ are homotopic, their induced morphisms on cohomologies are the same, i.e., $\mathrm{H}^j(F(f^\bullet \circ g^\bullet)) = \mathrm{H}^j(F1_{I^\bullet})$. For functors H^j and F, $\mathrm{H}^j(F1_{I^\bullet}) = 1_{\mathrm{H}^j(FI^\bullet)}$, i.e., the top equation of (7.4). Similarly, $\mathrm{H}^j(F(g^\bullet \circ f^\bullet)) = 1_{\mathrm{H}^j(FJ^\bullet)}$, the bottom equation of (7.4). Our goal is to prove $F(f^\bullet \circ g^\bullet) \sim F1_{I^\bullet}$ and $F(g^\bullet \circ f^\bullet) \sim F1_{J^\bullet}$. Notice that in general for a homotopy equivalence $f_1 \sim f_2$ (namely, $f_1 - f_2 = {}'\mathrm{d} \circ s + s \circ \mathrm{d}$ by the definition (3.2)), we have $F(f_1 - f_2) = Ff_1 - Ff_2 = F({}'\mathrm{d} \circ s + s \circ \mathrm{d}) = F'\mathrm{d} \circ Fs + Fs \circ F\mathrm{d}$ where F is an additive functor. That is $f_1 \sim f_2$ implies $Ff_1 \sim Ff_2$. In our case $f^\bullet \circ g^\bullet \sim 1_{I^\bullet}$ implies $F(f^\bullet \circ g^\bullet) \sim F1_{I^\bullet}$. Let $h^i = 1_{I^j} - f^j \circ g^j$, $j = 0, 1, 2, \ldots$. Consider the following diagram.



(7.5)

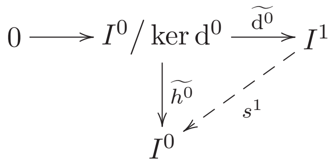
For the natural epimorphism $I^0 \xrightarrow{\pi^0} I^0 \big/ \ker \mathrm{d}^0$ we get the isomorphism

$$I^0 \big/ \ker \mathrm{d}^0 \xrightarrow{\approx} \operatorname{im} \mathrm{d}^0$$

in the abelian category $\mathscr{A}$. That is, we have the monomorphism

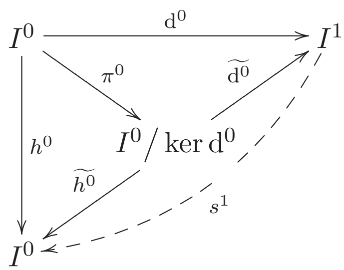
$$I^0 \big/ \ker \mathrm{d}^0 \approx \operatorname{im} \mathrm{d}^0 \xrightarrow{\widetilde{\mathrm{d}^0}} I^1,$$

i.e.,



(7.6)

where $\widetilde{h^0}(\overline{x^0}) = h^0(x^0)$ for $\overline{x^0} = \pi^0(x^0)$, $x^0 \in I^0$. Since I^0 is injective there exists a morphism $s^1 : I^1 \to I^0$ as shown in (7.6). In the diagram



(7.7)

we have

$$h^0 = 1_{I^0} - f^0 \circ g^0 = \widetilde{h^0} \circ \pi^0 = (s^1 \circ \widetilde{\mathrm{d}^0}) \circ \pi^0 = s^1 \circ (\widetilde{\mathrm{d}^0} \circ \pi^0) = s^1 \circ \mathrm{d}^0.$$

As in (7.5), since $I^{-1} = I^{-2} = \cdots = 0$, $1_{I^0} - f^0 \circ g^0 = s^1 \circ \mathrm{d}^0$ is (3.2) for $j = 0$. Namely, 1_{I^0} is homotopic to $f^0 \circ g^0$. In order to get $s^2 : I^2 \to I^1$ we need to be more careful since for the monomorphism

$$0 \longrightarrow I^1 \big/ \ker \mathrm{d}^1 \xrightarrow{\widetilde{\mathrm{d}^1}} I^2,$$

$\widetilde{h^1} : I^1 \big/ \ker \mathrm{d}^1 \to I^1$ will not become a morphism. As for $\widetilde{h^0}$, since the top sequence in (7.5) is exact, we have $\operatorname{im} \epsilon = \ker \mathrm{d}^0$ and $I^0 \big/ \ker \mathrm{d}^0 \approx I^0 \big/ \operatorname{im} \epsilon$. Then

$$\begin{aligned} \widetilde{h^0}(\overline{\epsilon(a)}) &= h^0(\epsilon(a)) = (1_{I^0} - f^0 \circ g^0)(\epsilon(a)) = \\ &= (1_{I^0} \circ \epsilon - (f^0 \circ g^0) \circ \epsilon)(a) = (\epsilon - \epsilon)(a) = 0. \end{aligned}$$

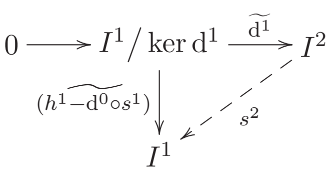
Therefore $\widetilde{h^0}(\overline{x^0}) = h^0(x^0)$ was fine. Namely, since $\widetilde{h^1}$ does not take $\mathrm{im}\,\mathrm{d}^0$ to zero, $\widetilde{h^1}$ will not be well-defined. For $x^0 \in I^0$ we need to define

$$(h^1 \widetilde{- \mathrm{d}^0 \circ} s^1) : I^1\big/ \ker \mathrm{d}^1 \; (\approx I^1\big/ \mathrm{im}\,\mathrm{d}^0) \longrightarrow I^1$$

as follows:

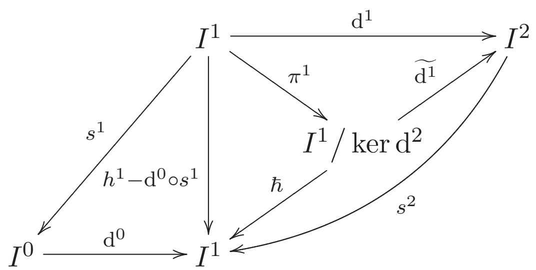
$$\begin{aligned}(h^1 \widetilde{- \mathrm{d}^0 \circ} s^1)(\overline{\mathrm{d}^0(x^0)}) &= (h^1 - \mathrm{d}^0 \circ s^1)(\mathrm{d}^0(x^0)) = \\ &= (h^1 \circ \mathrm{d}^0)(x^0) - (\mathrm{d}^0 \circ s^1 \circ \mathrm{d}^0)(x^0) = \\ &= ((1_{I^1} - f^1 \circ g^1) \circ \mathrm{d}^0)(x^0) - \\ &\qquad - \mathrm{d}^0 \circ (1_{I^0} - f^0 \circ g^0)(x^0) = \\ &= (1_{I^1} \circ \mathrm{d}^0 - d^0 \circ 1_{I^0})(x^0) + \\ &\qquad + (\mathrm{d}^0 \circ (f^0 \circ g^0) - (f^1 \circ g^1) \circ \mathrm{d}^0)(x^0) = \\ &= 0.\end{aligned}$$

For the diagram



(7.8)

we obtain $s^2 : I^2 \to I^1$. Then from the diagram



(7.9)

where we have put $\hbar = (h^1 \widetilde{- \mathrm{d}^0 \circ} s^1)$ to simplify readability of the diagram, we have

$$h^1 - \mathrm{d}^0 \circ s^1 = (h^1 \widetilde{- \mathrm{d}^0 \circ} s^1) \circ \pi^1 = (s^2 \circ \widetilde{\mathrm{d}^1}) \circ \pi^1 = s^2 \circ (\widetilde{\mathrm{d}^1} \circ \pi^1) = s^2 \circ \mathrm{d}^1,$$

obtaining $h^1 = \mathrm{d}^0 \circ s^1 + s^2 \circ \mathrm{d}^1$. That is,

$$1_{I^1} - f^1 \circ g^1 = \mathrm{d}^0 \circ s^1 + s^2 \circ \mathrm{d}^1.$$

Consequently, 1_{I^1} is homotopic to $f^1 \circ g^1$. The above method is valid for general j proving $1_{I^\bullet} \sim f^\bullet \circ g^\bullet$. As noted earlier, for an additive functor F, $F(f^\bullet \circ g^\bullet)$ is homotopic to $F1_{I^\bullet}$. Then we get $\mathrm{H}^j(F(f^\bullet \circ g^\bullet)) = \mathrm{H}^j(F1_{I^\bullet})$ which was to be proved, i.e., (7.4). For injective resolutions $I^\bullet$ and $J^\bullet$ of an object $A \in \mathrm{Ob}(\mathscr{A})$ we have an isomorphism $\mathrm{H}^j(FI^\bullet) \approx \mathrm{H}^j(FJ^\bullet)$. The isomorphic object R^jFA is the j-th right derived functor of F at A (see (7.1)).

Remark 16. Let $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ be a covariant left exact functor. Namely, for a short exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathscr{A}$, we have the exact sequence

$$0 \longrightarrow FA' \longrightarrow FA \longrightarrow FA''$$

in $\mathscr{B}$. Then as the contravariant functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}^\circ$, F becomes right exact. As a covariant functor $F : \mathscr{A}^\circ \rightsquigarrow \mathscr{B}^\circ$, F becomes right exact as well. As noted in Section 2.5, an injective resolution of A in $\mathscr{A}$ is a projective resolution of A in $\mathscr{A}^\circ$. Let $P_\bullet = I^\bullet$ be the projective resolution of A in $\mathscr{A}^\circ$, and let $P_\bullet \to A$, where $\eta = \epsilon^\circ : P_0 \to A \to 0$ in $\mathscr{A}^\circ$ for $0 \to A \xrightarrow{\epsilon} I^0$ in $\mathscr{A}$. Then $F : \mathscr{A}^\circ \rightsquigarrow \mathscr{B}^\circ$ induces the complex $FP_\bullet \to FA$:

$$\begin{array}{c} \cdots \longrightarrow FP_{j+1} \xrightarrow{F\mathrm{d}_{j+1}} FP_j \xrightarrow{F\mathrm{d}_j} FP_{j-1} \xrightarrow{F\mathrm{d}_{j-1}} \cdots \longrightarrow FP_0 \longrightarrow 0 \longrightarrow \cdots \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow F\eta \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad FA \end{array} \tag{7.10}$$

The subquotient object

$$\ker F\mathrm{d}_j \big/ \operatorname{im} F\mathrm{d}_{j+1} \tag{7.11}$$

is said to be the *j-th left derived functor* of F at A denoted as L_jFA. See the following:

$$\begin{array}{ccc} \mathscr{A} & \overset{F}{\rightsquigarrow} & \mathscr{B} \\ \wr\downarrow\circ & & \wr\downarrow\circ \\ \mathscr{A}^\circ & \overset{F}{\rightsquigarrow} & \mathscr{B}^\circ \end{array} \qquad\qquad \begin{array}{ccc} \mathscr{A} & \overset{\mathrm{R}^jF}{\rightsquigarrow} & \mathscr{B} \\ \wr\downarrow\circ & & \wr\downarrow\circ \\ \mathscr{A}^\circ & \overset{\mathrm{L}_jF}{\rightsquigarrow} & \mathscr{B}^\circ \end{array} \tag{7.12}$$

Those subquotient objects R^jFA and L_jFA, $j = 0, 1, 2, \ldots$, are generally referred to as the cohomologies and homologies of F at A, respectively. That is, for a complex $C^\bullet$ in an abelian category

$$\cdots \longrightarrow C^{-j} \xrightarrow{\mathrm{d}^{-j}} C^{-j+1} \longrightarrow \cdots \xrightarrow{\mathrm{d}^{-2}} C^{-1} \xrightarrow{\mathrm{d}^{-1}} C^0 \xrightarrow{\mathrm{d}^0}$$

$$\xrightarrow{\mathrm{d}^0} C^1 \longrightarrow \cdots \longrightarrow C^j \longrightarrow C^{j+1} \longrightarrow \cdots$$

the subquotient $\ker \mathrm{d}^j \big/ \operatorname{im} \mathrm{d}^{j-1}$, $j \geq 0$, is said to be the j-th cohomology of the complex, and the subquotient $\ker \mathrm{d}^{-j} \big/ \operatorname{im} \mathrm{d}^{-j-1}$ is the j-the homology of $C^\bullet$. Namely, $\mathrm{H}^{-j}(C^\bullet) = \mathrm{H}_j(C^\bullet)$ and $\mathrm{H}^j(C^\bullet) = \mathrm{H}_{-j}(C^\bullet)$.

2.8 Properties of Derived Functors

For an additive left exact functor F of abelian categories $\mathscr{A}$ and $\mathscr{B}$, i.e., $F : \mathscr{A} \rightsquigarrow \mathscr{B}$, we defined the derived functor $\mathrm{R}^j F : \mathscr{A} \rightsquigarrow \mathscr{B}$, $j = 0, 1, 2, \ldots$, in Section 2.7. We will compute $\mathrm{R}^0 FA$. Namely, for an injective resolution of A, $0 \to A \xrightarrow{\epsilon} I^0 \xrightarrow{\mathrm{d}^0} I^1 \xrightarrow{\mathrm{d}^1} \ldots$, $\mathrm{R}^0 FA$ is the 0-th cohomology of the complex $FI^\bullet$, i.e., $\mathrm{H}^0(FI^\bullet) = \ker F\mathrm{d}^0$. Since F is left exact,

$$0 \longrightarrow FA \xrightarrow{F\epsilon} FI^0 \xrightarrow{F\mathrm{d}^0} F \operatorname{im} \mathrm{d}^0$$

is exact sequence. Then $\ker F\mathrm{d}^0 = \operatorname{im} F\epsilon$ and $F\epsilon$ is a monomorphism. Consequently, we have $\ker F\mathrm{d}^0 = \operatorname{im} F\epsilon \approx FA$ for any object A of $\mathscr{A}$. Therefore we obtain,

(D.F.0) $$\mathrm{R}^0 F \approx F.$$

For a morphism $f : A \to B$ in $\mathscr{A}$ we have the induced morphism

$$\mathrm{R}^j F : \mathrm{R}^j FA \to \mathrm{R}^j FB$$

in $\mathscr{B}$. Namely, $\mathrm{R}^j F$ is actually a functor. This means that there are injective resolutions $I^\bullet$ and $J^\bullet$ of A and B, respectively, so that

$$\begin{array}{ccc} I^j & \xrightarrow{\mathrm{d}^j} & I^{j+1} \\ \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{j+1}} \\ J^j & \xrightarrow{'\mathrm{d}^j} & J^{j+1} \end{array} \tag{8.1}$$

may be commutative for all $j = 0, 1, 2, \ldots$. Furthermore, for a short exact sequence

$$0 \longrightarrow A' \xrightarrow{\phi} A \xrightarrow{\psi} A'' \longrightarrow 0$$

we have not only

$$\mathrm{R}^j FA' \xrightarrow{\mathrm{R}^j F\phi} \mathrm{R}^j FA \xrightarrow{\mathrm{R}^j F\psi} \mathrm{R}^j FA'',$$

but also $\partial^j : \mathrm{R}^j FA'' \to \mathrm{R}^{j+1} FA'$ so that

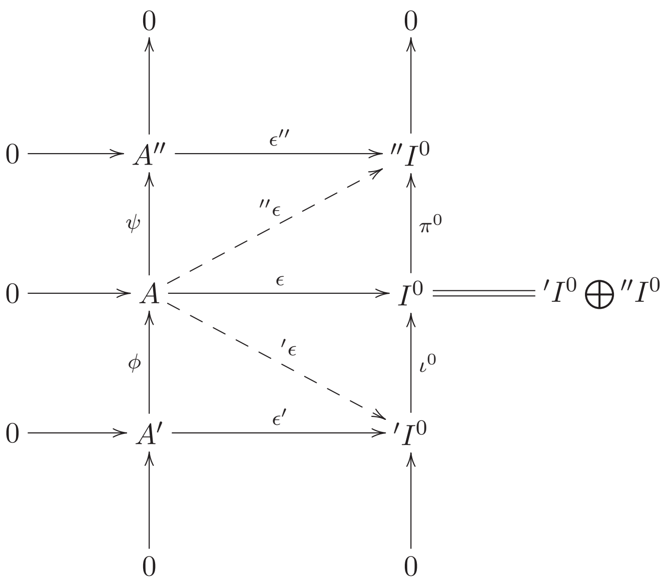
(D.F.1)

$$0 \longrightarrow \mathrm{R}^0FA' \xrightarrow{\mathrm{R}^0F\phi} \mathrm{R}^0FA \xrightarrow{\mathrm{R}^0F\psi} \mathrm{R}^0FA'' \xrightarrow{\partial^0} \mathrm{R}^1FA' \xrightarrow{\mathrm{R}^1F\phi} \cdots$$

$$\cdots \xrightarrow{\partial^{j-1}} \mathrm{R}^jFA' \xrightarrow{\mathrm{R}^jF\phi} \mathrm{R}^jFA \xrightarrow{\mathrm{R}^jF\psi} \mathrm{R}^jFA'' \xrightarrow{\partial^j} \mathrm{R}^{j+1}FA' \longrightarrow \cdots$$

may be an exact sequence in $\mathscr{B}$.

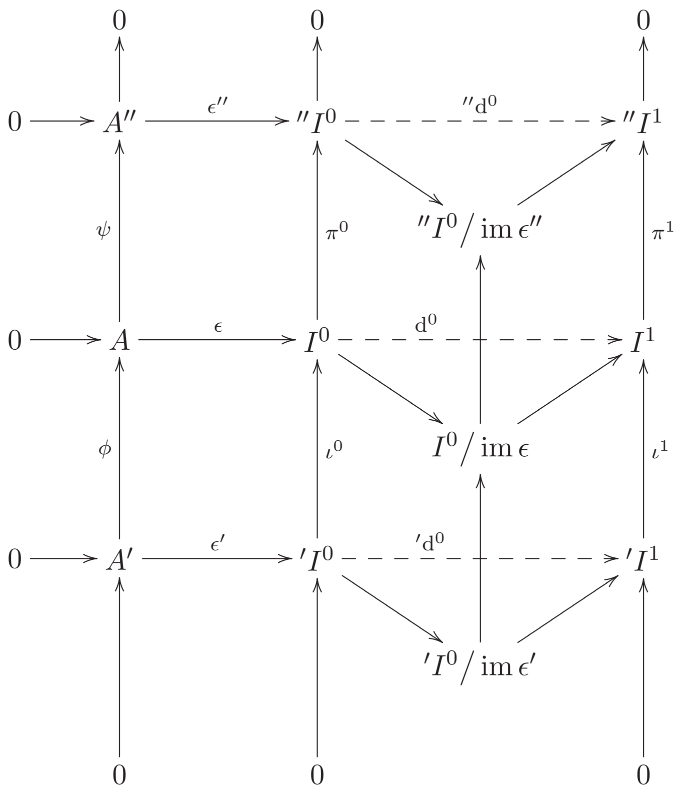
A proof of (D.F.1) can be done as follows. For A' and A'' let ϵ' and ϵ'' be monomorphisms into injective objects $'I^0$ and $''I^0$ as in Section 2.6 (i.e., the initial terms of injective resolutions for A' and A''). Then let $I^0 := {}'I^0 \oplus {}''I^0$ to obtain



(8.2)

where $\iota^0 : {}'I^0 \to I^0 = {}'I^0 \oplus {}''I^0$ is defined by $\iota^0(x') = (x', 0) \in {}'I^0 \oplus {}''I^0$ and $\pi^0 : I^0 = {}'I^0 \oplus {}''I^0 \to {}''I^0$ is the projection defined by $\pi^0(x', x'') = x''$. Then ι^0 is a monomorphism and π^0 is an epimorphism satisfying $\ker \pi^0 = \operatorname{im} \iota^0$. Next we will show that there is a monomorphism $\epsilon : A \to I^0$. For $0 \to A' \xrightarrow{\phi} A$ and $\epsilon' : A' \to {}'I^0$ there exists a morphism $'\epsilon : A \to {}'I^0$ satisfying $\epsilon' = {}'\epsilon \circ \phi$ (i.e., (5.4)). Let $''\epsilon = \epsilon'' \circ \psi : A \to {}''I^0$. Then define $\epsilon : A \to I^0 = {}'I^0 \oplus {}''I^0$ by $\epsilon(a) = ({}'\epsilon(a), {}''\epsilon(a))$ for $a \in A$, obtaining the commutative diagram (8.2).

Next consider the following diagram.



(8.3)

As we constructed $0 \to {}'I^0 \to I^0 \to {}''I^0$ for $0 \to A' \to A \to A'' \to 0$, for the short exact sequence

$$0 \longrightarrow {}'I^0 \big/ \operatorname{im}\epsilon' \longrightarrow I^0 \big/ \operatorname{im}\epsilon \longrightarrow {}''I^0 \big/ \operatorname{im}\epsilon'' \longrightarrow 0 \tag{8.4}$$

in the third column of (8.3), we obtain $0 \to {}'I^1 \to I^1 \to {}''I^1 \to 0$ as shown in the fourth column of (8.3). We define ${}'\mathrm{d}^0$, d^0 and ${}''\mathrm{d}^0$ as the compositions ${}'I^0 \to {}'I^0 \big/ \operatorname{im}\epsilon' \to {}'I^1$, $I^0 \to I^0 \big/ \operatorname{im}\epsilon \to I^1$ and ${}''I^0 \to {}''I^0 \big/ \operatorname{im}\epsilon'' \to {}''I^1$, respectively. Thus we can obtain the exact splitting sequence of complexes

$$0 \longrightarrow {}'I^\bullet \xrightarrow{\iota^\bullet} I^\bullet \xrightarrow{\pi^\bullet} {}''I^\bullet \longrightarrow 0 \tag{8.5}$$

which are injective resolutions of A', A and A'', respectively. Therefore, we obtain the exact sequence of complexes

$$0 \longrightarrow F'I^\bullet \xrightarrow{F\iota^\bullet} FI^\bullet \xrightarrow{F\pi^\bullet} {}''FI^\bullet \longrightarrow 0. \tag{8.6}$$

By taking cohomologies of (8.6), we get

$$
\begin{array}{l}
\overset{\partial^{j-1}}{\dashrightarrow} \mathrm{H}^j(F'I^\bullet) \longrightarrow \mathrm{H}^j(FI^\bullet) \longrightarrow \mathrm{H}^j(F''I^\bullet) \overset{\partial^j}{\dashrightarrow} \\
\overset{\partial^j}{\dashrightarrow} \mathrm{H}^{j+1}(F'I^\bullet) \longrightarrow \cdots
\end{array}
\tag{8.7}
$$

We wish to define the connecting morphism ∂^j in (8.7) from

$$\mathrm{H}^j(F''I^\bullet) = \mathrm{R}^j FA'' \longrightarrow \mathrm{H}^{j+1}(F'I^\bullet) = \mathrm{R}^{j+1}FA'$$

(in the long exact sequence (D.F.1)). In the commutative diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
\cdots & \xrightarrow{F''\mathrm{d}^{j-1}} & F''I^j & \xrightarrow{F''\mathrm{d}^j} & F''I^{j+1} & \xrightarrow{F''\mathrm{d}^{j+1}} & F''I^{j+2} & \longrightarrow \cdots : & F''I^\bullet \\
 & & \uparrow{\scriptstyle F\pi^j} & & \uparrow{\scriptstyle F\pi^{j+1}} & & \uparrow & & \uparrow{\scriptstyle F\pi^\bullet} \\
\cdots & \xrightarrow{F\mathrm{d}^{j-1}} & FI^j & \xrightarrow{F\mathrm{d}^j} & FI^{j+1} & \xrightarrow{F\mathrm{d}^{j+1}} & FI^{j+2} & \longrightarrow \cdots : & FI^\bullet \\
 & & \uparrow{\scriptstyle F\iota^j} & & \uparrow & & \uparrow{\scriptstyle F\iota^{j+1}} & & \uparrow{\scriptstyle F\iota^\bullet} \\
\cdots & \xrightarrow{F'\mathrm{d}^{j-1}} & F'I^j & \xrightarrow{F'\mathrm{d}^j} & F'I^{j+1} & \xrightarrow{F'\mathrm{d}^{j+1}} & F'I^{j+2} & \longrightarrow \cdots : & F'I^\bullet \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0 & & 0
\end{array}
\tag{8.8}
$$

let

$$\overline{''y^j} \in \mathrm{R}^j FA'' = \mathrm{H}^j(F''I^\bullet) = \ker F''\mathrm{d}^j \big/ \operatorname{im} F''\mathrm{d}^{j-1}$$

where $''y^j \in \ker F''\mathrm{d}^j$. Since $F\pi^j$ is epimorphic, there is a $y^j \in FI^j$ satisfying $F\pi^j(y^j) = {''y^j}$. Then $F\pi^{j+1}(F\mathrm{d}^j(y^j)) \in F''I^{j+1}$ equals

$$F''\mathrm{d}^j(F\pi^j(y^j)) = F''\mathrm{d}^j(''y^j) = 0.$$

Namely, $F\mathrm{d}^j(y^j) \in \ker F\pi^{j+1}$. The exactness of the second column implies that $F\mathrm{d}^j(y^j) = F\iota^{j+1}('y^{j+1})$ for some $'y^{j+1} \in F'I^{j+1}$. Having obtained $'y^{j+1}$ in $F'I^{j+1}$, first we need to confirm $'y^{j+1} \in \ker F'\mathrm{d}^{j+1}$ to get the cohomological class $\overline{'y^{j+1}} \in \mathrm{R}^{j+1}FA' = \mathrm{H}^{j+1}(F'I^\bullet)$. The commutativity of the lower right-hand-side square of (8.8) implies

$$
\begin{aligned}
F\iota^{j+2}(F'\mathrm{d}^{j+1}('y^{j+1})) &= F\mathrm{d}^{j+1}(F\iota^{j+1}('y^{j+1})) = F\mathrm{d}^{j+1}(F\mathrm{d}^j(y^j)) = \\
&= (F\mathrm{d}^{j+1} \circ F\mathrm{d}^j)(y^j) = F(\mathrm{d}^{j+1} \circ \mathrm{d}^j)(y^j) = 0.
\end{aligned}
$$

Since $F\iota^{j+2}$ is a monomorphism, we have $F'\mathrm{d}^{j+1}({}'y^{j+1}) = 0$, i.e., we have ${}'y^{j+1} \in \ker F'\mathrm{d}^{j+1}$, inducing, $\overline{{}'y^{j+1}} \in \mathrm{H}^{j+1}(F'I^\bullet) = \mathrm{R}^{j+1}FA'$. Then we define the connecting morphism $\partial^j : \mathrm{R}^jFA'' \to \mathrm{R}^{j+1}FA'$ as follows:

$$\partial^j({}''y^j) = \overline{{}'y^{j+1}}. \tag{8.9}$$

One may wish to check that the definition of ∂^j in (8.9) is well-defined.

Lastly, as an exercise, the reader may want to prove the exactness of the long sequence of (D.F.1). Namely, (D.F.1) is exact if $\operatorname{im}\mathrm{R}^jF\phi = \ker \mathrm{R}^jF\psi$, $\operatorname{im}\mathrm{R}^jF\psi = \ker\partial^j$ and $\operatorname{im}\partial^j = \ker \mathrm{R}^{j+1}F\phi$, $j = 0, 1, 2, \ldots$. Since we have $\mathrm{R}^0F \approx F$, which is left exact, the case $j = 0$:

$$0 \to \mathrm{R}^0FA' \to \mathrm{R}^0FA \to \mathrm{R}^0FA''$$

is exact. Furthermore, since R^jF is a functor, $\mathrm{R}^jF\psi \circ \mathrm{R}^jF\phi = \mathrm{R}^jF(\psi\circ\phi) = 0$ holds, i.e.,

$$\operatorname{im}\mathrm{R}^jF\phi \subset \ker \mathrm{R}^jF\psi.$$

The remaining portion on the exactness can be proved by "diagram chasing".

Next we will prove the third property (D.F.2) of derived functors. Let us consider the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \xrightarrow{\phi} & A & \xrightarrow{\psi} & A'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \\
0 & \longrightarrow & B' & \xrightarrow{\lambda} & B & \xrightarrow{\mu} & B'' & \longrightarrow & 0
\end{array} \tag{8.10}$$

From (D.F.1), we have two horizontal long exact sequences

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & \mathrm{R}^jFA' & \xrightarrow{\mathrm{R}^jF\phi} & \mathrm{R}^jFA & \xrightarrow{\mathrm{R}^jF\psi} & \mathrm{R}^jFA'' & \xrightarrow{\partial^j} & \mathrm{R}^{j+1}FA' & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle \mathrm{R}^jFf} & & \downarrow{\scriptstyle \mathrm{R}^jFg} & & \downarrow{\scriptstyle \mathrm{R}^jFh} & & \downarrow{\scriptstyle \mathrm{R}^{j+1}Ff} & & \\
\cdots & \longrightarrow & \mathrm{R}^jFB' & \xrightarrow{\mathrm{R}^jF\lambda} & \mathrm{R}^jFB & \xrightarrow{\mathrm{R}^jF\mu} & \mathrm{R}^jFB'' & \xrightarrow{\delta^j} & \mathrm{R}^{j+1}FB' & \longrightarrow & \cdots
\end{array} \tag{8.11}$$

with the induced vertical morphisms. The commutativity of the first two squares comes from commutativity of the diagram (8.10).

(D.F.2) We will prove that the third square of (8.11)

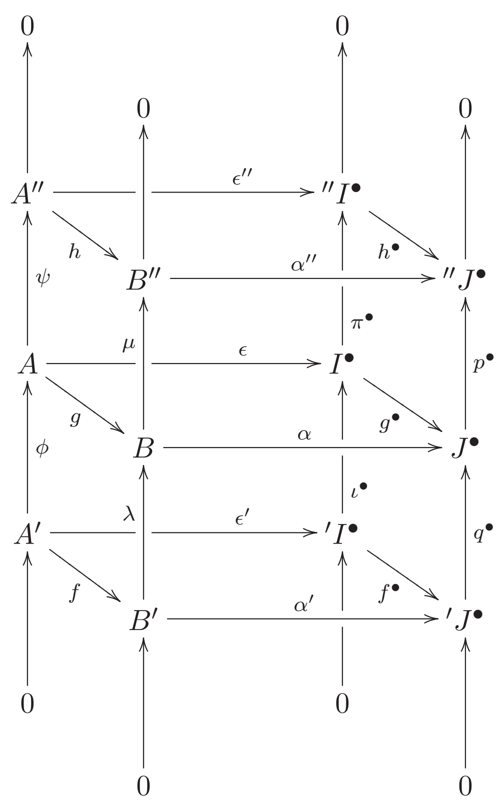
$$\begin{array}{ccc}
\mathrm{R}^jFA'' & \xrightarrow{\partial^j} & \mathrm{R}^{j+1}FA' \\
\downarrow{\scriptstyle \mathrm{R}^jFh} & & \downarrow{\scriptstyle \mathrm{R}^{j+1}Ff} \\
\mathrm{R}^jFB'' & \xrightarrow{\delta^j} & \mathrm{R}^{j+1}FB'
\end{array}$$

is commutative. We will give a direct proof (D.F.2) as follows.

Proof of (D.F.2). Let $'I^\bullet$, $I^\bullet$ and $''I^\bullet$ be injective resolutions of A', A and A'' and let $'J^\bullet$, $J^\bullet$ and $''J^\bullet$ be injective resolutions of B', B and B'', respectively, as constructed in (8.5), so that the diagram

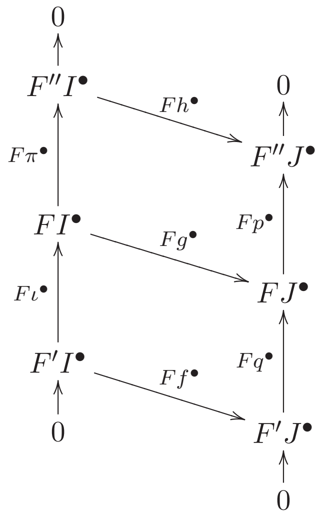
$$\begin{array}{ccccccccc} 0 & \longrightarrow & 'I^\bullet & \xrightarrow{\iota^\bullet} & I^\bullet & \xrightarrow{\pi^\bullet} & ''I^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f^\bullet} & & \downarrow{\scriptstyle g^\bullet} & & \downarrow{\scriptstyle h^\bullet} & & \\ 0 & \longrightarrow & 'J^\bullet & \xrightarrow{q^\bullet} & J^\bullet & \xrightarrow{p^\bullet} & ''sJ^\bullet & \longrightarrow & 0 \end{array} \tag{8.12}$$

becomes commutative. That is, we have the following diagram:



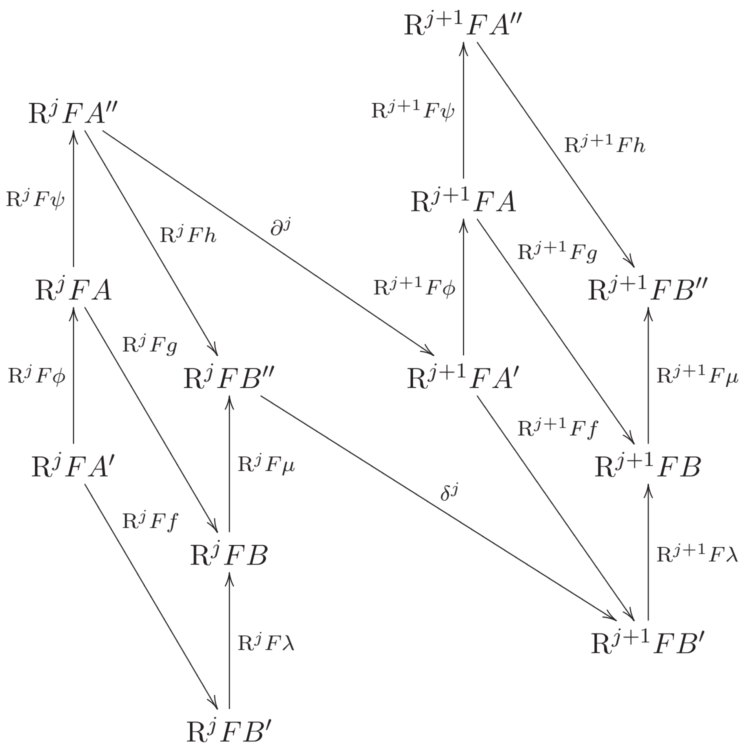
(8.13)

Then the functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ induces the commutative diagram of complexes.



(8.14)

Then, by taking cohomologies of (8.14), the commutative diagram (8.12) can be written



(8.15)

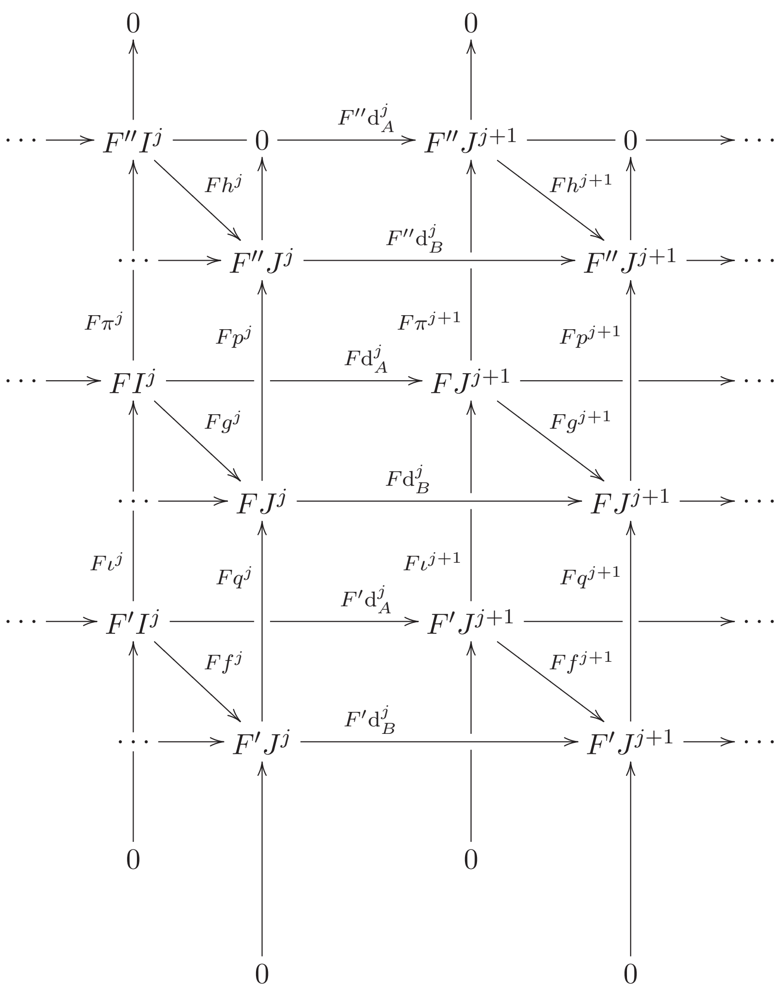
Notice that the commutativity of (D.F.2) is the commutativity of the rhombus in the middle of (8.15). That is, we will prove that

$$(\mathrm{R}^{j+1}Ff \circ \partial^j)(\overline{''y^j}) = (\delta^j \circ \mathrm{R}^j Fh)(\overline{''y^j}), \tag{8.16}$$

where $\overline{''y^j} \in \mathrm{R}^j FA''$ and

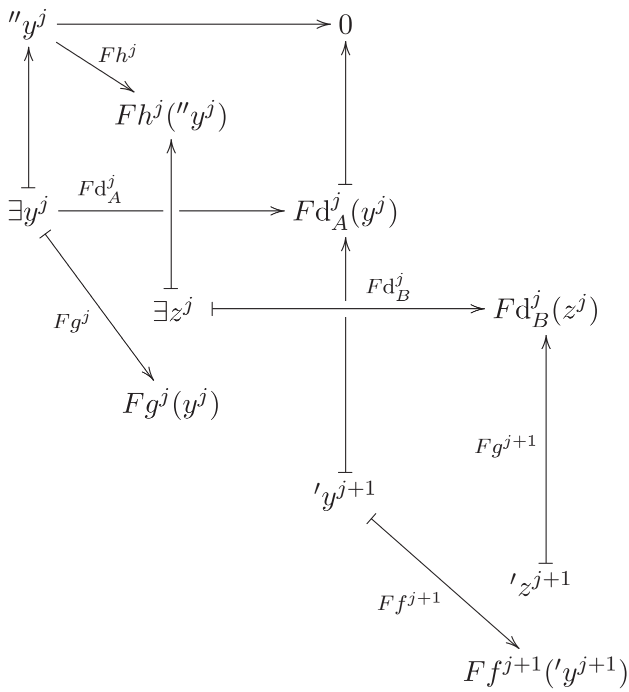
$$''y^j \in \ker(F''\mathrm{d}^j : F''I^j \xrightarrow{F''\mathrm{d}^j} F''I^{j+1}).$$

The right side of (8.16) is $\delta^j(\overline{Fh^j(''y^j)})$, and the left side is $\mathrm{R}^{j+1}Ff(\overline{'y^j})$ as defined in (8.9). From the following diagram at degrees j and $j+1$ of (8.14)



(8.17)

For $''y^j \in F''I^j$, the diagram chasing in (8.17) becomes



(8.18)

The cohomology class on the the left hand-side of (8.16) is determined by

$$Ff^{j+1}('y^{j+1}) \in F'J^{j+1},$$

and the cohomology class on the right hand-side of (8.16) is determined by $'z^{j+1} \in F'J^{j+1}$. In order to prove (8.16) we need to show

$$'z^{j+1} - Ff^{j+1}('y^{j+1}) \in \operatorname{im} F\mathrm{d}_B^j.$$

Notice that $'z^{j+1} - Ff^{j+1}('y^{j+1})$ is an element of $F'J^{j+1}$ at the lower right of (8.17). Then Fq^{j+1} takes this element to

$$Fq^{j+1}('z^{j+1}) - Fq^{j+1}(Ff^{j+1}('y^{j+1})) = F\mathrm{d}_B^j(z^j) - Fg^{j+1}(F\iota^{j+1}('y^{j+1}))$$

in FJ^{j+1}. Since $F\iota^{j+1}('y^{j+1}) = F\mathrm{d}_A^j(y^j)$, the above equation can be continued as

$$\begin{aligned} Fq^{j+1}('z^{j+1}) - Fq^{j+1}(Ff^{j+1}('y^{j+1})) &= F\mathrm{d}_B^j(z^j) - Fg^{j+1}(F\iota^{j+1}('y^{j+1})) \\ &= F\mathrm{d}_B^j(z^j) - Fg^{j+1}(F\mathrm{d}_A^j(y^j)). \end{aligned}$$

Furthermore, by the commutativity of the middle square of (8.17), we can can continue as

$$F\mathrm{d}_B^j(z') - F\mathrm{d}_B^j(Fg^j(y^j)) = F\mathrm{d}_B^j(z^j - Fg^j(y^j)).$$

For $z^j - Fg^j(y^j) \in FJ^j$, the map Fp^j takes this element to

$$\begin{aligned} Fp^j(z^j - Fg^j(y^j)) &= Fh^j({''y^j}) - Fp^j(Fg^j(y^j)) \\ &= Fh^j({''y^j}) - Fh^j(F\pi^j(y^j)). \end{aligned}$$

Since $F\pi^j(y^j) = {''y^j}$, the above equals $Fh^j({''y^j}) - Fh^j(F\pi^j(y^j)) = 0$. Namely, we get $z^j - Fg^j(y^j) \in \ker Fp^j$. Since (8.17) is vertically exact, there exists ${'z^j} \in F'J^j$ satisfying $Fq^j({'z^j}) = z^j - Fg^j(y^j)$. In order to prove (8.16), we need to show that ${'z^{j+1}}$ and $Ff^{j+1}({'y^{j+1}})$ belong to the same cohomology class. Namely, we must show

$$F'\mathrm{d}_B^j({'z^j}) = {'z^{j+1}} - Ff^{j+1}({'y^{j+1}}). \tag{8.19}$$

Since Fq^{j+1} is a monomorphism, instead of (8.19), it is enough to show

$$Fq^{j+1}(F'\mathrm{d}_B^j({'z^j}) - {'z^{j+1}} + Ff^{j+1}({'y^{j+1}})) = 0. \tag{8.20}$$

We will compute the left side of (8.20) as follows

$$Fq^{j+1}(F'\mathrm{d}_B^j({'z^j})) - Fq^{j+1}({'z^{j+1}}) + Fq^{j+1}(Ff^{j+1}({'y^{j+1}})),$$

which (since $Fq^j({'z^j}) = z^j - Fg^j(y^j) \in \ker Fp^j$) equals

$$\begin{aligned} &F\mathrm{d}_B^j(Fq^j({'z^j})) - F\mathrm{d}_B^j(z^j) + Fg^{j+1}(F\mathrm{d}_A^j(y^j)) \\ &\quad = F\mathrm{d}_B^j(z^j - Fg^j(y^j)) - F\mathrm{d}_B^j(z^j) + F\mathrm{d}_B^j(Fg^j(y^j)) = 0 \end{aligned}$$

by the commutativity of the middle square of (8.17). Consequently (D.F.2) is proved.

Property (D.F.3)

Let I be an injective object of $\mathscr{A}$. then we can consider the following trivial resolution of I:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & I & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & \downarrow & & \downarrow{\scriptstyle \epsilon} & & \downarrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & I & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{8.21}$$

By the definition

$$\begin{aligned} \mathrm{R}^j FI &= \mathrm{H}^j(\cdots \longrightarrow 0 \longrightarrow FI \longrightarrow 0 \longrightarrow \cdots) = \\ &= \begin{cases} 0 & \text{for } j = 1, 2, \ldots \\ FI & \text{for } j = 0. \end{cases} \end{aligned}$$

Namely, for an injective object I

(D.F.3) $\mathrm{R}^j FI = 0$ for $j = 1, 2, \ldots$.

2.9 Axioms for Derived Functors

The properties (D.F.0) through (D.F.3) of derived functors $(\mathrm{R}^j F)_{j\geq 0}$ of a left exact additive functor F of abelian categories $\mathscr{A}$ and $\mathscr{B}$ can be used as the characterization of derived functors. For a positive integer j, let T^j be an additive functor from an abelian category $\mathscr{A}$ to another abelian category $\mathscr{B}$. Then the sequence $(T^j)_{j\geq 0}$ of functors is said to an *exact connected sequence of functors* from $\mathscr{A}$ to $\mathscr{B}$ if (D.F.1) and (D.F.2) are satisfied for $(T^j)_{j\geq o}$ (i.e., replace $\mathrm{R}^j F$ by T^j in (D.F.1) and (D.F.2)). Let $\mathscr{C} := \mathscr{B}^{\mathscr{A}}$. Let $\mathsf{CSe}(\mathscr{C})$ and $\mathsf{ECSe}(\mathscr{C})$ be the category of connected sequences (i.e., without the exactness of (D.F.1) in $\mathscr{B}$) and the category of exact connected sequences of functors, respectively. A morphism $f^* = (f^j)_{j\geq 0}$ between objects $T^* := (T^j)_{j\geq 0}$ and $S^* := (S^j)_{j\geq 0}$ of $\mathsf{CSe}(\mathscr{C})$ (or $\mathsf{ECSe}(\mathscr{C})$) is a sequence of natural transformations $f^j : T^j \to S^j$, $j \geq 0$. Then T^* is said to be universal in $\mathsf{CSe}(\mathscr{C})$ when the following condition is satisfied.

(UCS) For an object S^* in $\mathsf{CSe}(\mathscr{C})$ and for a natural transformation $h : T^0 \to S^0$, there exists a unique morphism f^* from T^* to S^* satisfying $f^0 = h$.

Let $'T^* := ('T^j)_{j\geq 0}$ be another object of $\mathsf{CSe}(\mathscr{C})$ and let $g^0 : {}'T^0 \to T^0$ be any natural transformation. From (UCS) there is a unique $f^* : {}'T^* \to T^*$ satisfying $f^0 = g^0$. By reversing the role we also get a unique $'f^* : T^* \to {}'T^*$ satisfying $'f^0 = {}'g^0 : T^0 \to {}'T^0$. Then we have $'f^* \circ f^* : {}'T^* \to {}'T^*$ and $f^* \circ {}'f^* : T^* \to T^*$ satisfying $'f^0 \circ f^0 = {}'g^0 \circ g^0 : {}'T^0 \to {}'T^0$ and $f^0 \circ {}'f^0 : T^0 \to T^0$. The uniqueness of the morphism in (UCS) and the identities 1_{T^*} and $1_{'T^*}$ being morphisms in $\mathsf{CSe}(\mathscr{C})$ imply that a universal connected sequence of functors T^* is determined by T^0 up to a canonical isomorphism. This universal object T^* is said to be the derived functors of T^0.

Let F be a left exact additive functor from an abelian category $\mathscr{A}$ with enough injectives to an abelian category $\mathscr{B}$. Consequently, if $T^* = (T^j)_{j\geq 0}$ is an exact connected sequence of functors from $\mathscr{A}$ to $\mathscr{B}$ satisfying (D.F.0), $T^0 \approx F$ and (D.F.3), $T^j(I) = 0$ for an injective object I of $\mathscr{A}$, $j \geq 1$, then T^* are the derived functors of F (i.e., of T^0). Namely, if $T^* = (T^j)_{j\geq 0}$ is a connected sequence of functors from $\mathscr{A}$ to $\mathscr{B}$ and if a natural transformation h^0 from $\mathrm{R}^0 F$ to T^0 is given, then there exists a unique morphism of connected sequences of functors

$$h^* = (h^j)_{j\geq 0} : \mathrm{R}^* F = (\mathrm{R}^j F)_{j\geq 0} \longrightarrow T^* = (T^j)_{j\geq 0} \tag{9.1}$$

so that we may have the commutative diagram

$$\begin{array}{ccc} \mathrm{R}^j F A'' & \xrightarrow{\partial^j} & \mathrm{R}^{j+1} F A' \\ \Big\downarrow{\scriptstyle h^j} & & \Big\downarrow{\scriptstyle h^{j+1}} \\ T^j A'' & \xrightarrow{\mathrm{d}^j} & T^{j+1} A'. \end{array} \tag{9.2}$$

A characterization of the derived functors of a left exact functor F from an abelian category $\mathscr{A}$ to an abelian category $\mathscr{B}$ is the following:

(1) A connected sequence of functors $(T^j)_{j\geq 0}$;

(2) There is a natural transformation $h : F \to T^0$;

(3) The universal property is satisfied, i.e., for another connected sequence of functors ${}'T^* := ({}'T^j)_{j\geq 0}$ and a natural transformation $g : F \to {}'T^0$ there exists a unique morphism $\lambda : T^0 \to {}'T^0$ so that

$$\begin{array}{ccc} F & \xrightarrow[h]{\approx} & T^0 \\ & {\scriptstyle g}\searrow \qquad \swarrow{\scriptstyle \lambda} & \\ & {}'T^0 & \end{array}$$

commutes.

2.10 The Derived Functors $(\mathrm{Ext}^j)_{j\geq 0}$

Let $\mathscr{A}$ be an abelian category with enough injectives. Recall from Section 1.6 that for A and B in $\mathrm{Ob}(\mathscr{A})$, $\mathrm{Hom}_{\mathscr{A}}(B, A)$ is an abelian group (condition (Ab.1)) and that $\mathrm{Hom}_{\mathscr{A}}(\cdot, A) : \mathscr{A} \rightsquigarrow \mathsf{Ab}$ is a left exact additive contravariant functor. For an injective object I, $\mathrm{Hom}_{\mathscr{A}}(\cdot, I)$ is an exact functor. (See Section 2.5.) The j-th derived functor $\mathrm{R}^j \mathrm{Hom}_{\mathscr{A}^\circ}(\cdot, A)$ of $\mathrm{Hom}_{\mathscr{A}^\circ}(\cdot, A) : \mathscr{A}^\circ \rightsquigarrow \mathsf{Ab}$ is defined by

$$\mathrm{R}^j \mathrm{Hom}_{\mathscr{A}^\circ}(\cdot, A)B = \mathrm{R}^j \mathrm{Hom}_{\mathscr{A}^\circ}(B, A) := \mathrm{H}^j(\mathrm{Hom}_{\mathscr{A}^\circ}(I^\bullet, A)) \tag{10.1}$$

where $I^\bullet$ is an injective resolution of B in $\mathscr{A}^\circ$ (i.e., $I^\bullet$ is a projective resolution of B in $\mathscr{A}$). The j-th derived functor $\mathrm{R}^j \mathrm{Hom}_{\mathscr{A}}(B, \cdot)$ of the left additive covariant functor $\mathrm{Hom}_{\mathscr{A}}(B, \cdot) : \mathscr{A} \rightsquigarrow \mathsf{Ab}$ is defined by

$$\mathrm{R}^j \mathrm{Hom}_{\mathscr{A}}(B, \cdot)A = \mathrm{R}^j \mathrm{Hom}_{\mathscr{A}}(B, A) := \mathrm{H}^j(\mathrm{Hom}_{\mathscr{A}}(B, J^\bullet)) \tag{10.2}$$

where $J^\bullet$ is an injective resolution of A in $\mathscr{A}$. Note that $\mathrm{Hom}_{\mathscr{A}}(B, J^\bullet)$ is the complex

$$\mathrm{Hom}_{\mathscr{A}}(B, J^0) \xrightarrow{{}'\mathrm{d}^0_*} \mathrm{Hom}_{\mathscr{A}}(B, J^1) \xrightarrow{{}'\mathrm{d}^1_*} \cdots \xrightarrow{{}'\mathrm{d}^{j-1}_*} \mathrm{Hom}_{\mathscr{A}}(B, J^j) \xrightarrow{{}'\mathrm{d}^j_*} \cdots \tag{10.3}$$

in Ab, and that $\mathrm{Hom}_{\mathscr{A}^\circ}(I^\bullet, A)$ is the complex

$$\cdots \xrightarrow{\mathrm{d}^{j\,*}} \mathrm{Hom}_{\mathscr{A}^\circ}(I^j, A) \xrightarrow{\mathrm{d}^{j-1\,*}} \cdots \xrightarrow{\mathrm{d}^{1\,*}} \mathrm{Hom}_{\mathscr{A}^\circ}(I^1, A) \xrightarrow{\mathrm{d}^{0\,*}} \mathrm{Hom}_{\mathscr{A}^\circ}(I^0, A), \tag{10.4}$$

or

$$\mathrm{Hom}_{\mathscr{A}}(P_0, A) \xrightarrow{\mathrm{d}_0^*} \mathrm{Hom}_{\mathscr{A}}(P_1, A) \xrightarrow{\mathrm{d}_1^*} \cdots \xrightarrow{\mathrm{d}_{j-1}^*} \mathrm{Hom}_{\mathscr{A}}(P_j, A) \xrightarrow{\mathrm{d}_j^*} \cdots \tag{10.5}$$

where $P_\bullet = I^\bullet$ as in Remark 15. Furthermore, one can consider the complex

$$(C^j)_{j\geq 0} := \Big(\bigoplus_{l+l'=j} \mathrm{Hom}_{\mathscr{A}}(P_l, I^{l'}) \Big)_{j\geq 0}$$

(see the Section on double complexes in Chapter III),

$$C^0 \xrightarrow{(d_0^*, {}'\mathrm{d}_*^0)} C^1 \xrightarrow{(d_1^*, {}'\mathrm{d}_*^1)} \cdots \xrightarrow{(d_{j-1}^*, {}'\mathrm{d}_*^{j-1})} C^j \xrightarrow{(d_j^*, {}'\mathrm{d}_*^j)} \cdots \tag{10.6}$$

Then we have

$$\begin{aligned} \mathrm{H}^j(\mathrm{Hom}_{\mathscr{A}}(B, J^\bullet)) \approx \mathrm{H}^j(\mathrm{Hom}_{\mathscr{A}^\circ}(I^\bullet, A)) = \mathrm{H}^j(\mathrm{Hom}_{\mathscr{A}}(P_\bullet, A)) \approx \\ \approx \mathrm{H}^j(C^\bullet). \end{aligned}$$

Namely, all the cohomology groups of the complexes are (10.3), (10.4), (10.5) and (10.6) are isomorphic to each other. See H. Cartan–S. Eilenberg, *Homological Algebra*, Princeton University Press, 1956, Chapter VI for a proof. This isomorphic object in Ab is written as $\mathrm{Ext}^j_{\mathscr{A}}(A, B)$, the j-th derived functor in the sense of (10.1) (and (10.2)). Consequently, $\mathrm{Ext}^j_{\mathscr{A}}(A, B)$ satisfies (D.F.0), i.e., for $j \geq 1$ $\mathrm{Ext}^0_{\mathscr{A}}(A, B) \approx \mathrm{Hom}_{\mathscr{A}}(A, B)$, (D.F.1), (D.F.2) and (D.F.3), i.e.,

$$\mathrm{Ext}^j_{\mathscr{A}}(B, I) = \mathrm{Ext}^j_{\mathscr{A}}(P, A) = \mathrm{Ext}^j_{\mathscr{A}^\circ}(I, A) = 0$$

for an injective object I and a projective object P.

2.11 Precohomology

For a complex $(C^\bullet, \mathrm{d}^\bullet)$ in an abelian category $\mathscr{A}$, the subquotient object $\mathrm{H}^j(C^\bullet) = \ker \mathrm{d}^j \big/ \mathrm{im}\, \mathrm{d}^{j-1}$, i.e., the j-th cohomology of $C^\bullet$ exists. For a sequence C^* of objects and morphisms in $\mathscr{A}$

$$\cdots \longrightarrow C^{j-1} \xrightarrow{\mathrm{d}^{j-1}} C^j \xrightarrow{\mathrm{d}^j} C^{j+1} \xrightarrow{\mathrm{d}^{j+1}} \cdots \tag{11.1}$$

which need not satisfy $\mathrm{d}^j \circ \mathrm{d}^{j-1} = 0$, we will define a new invariant as a generalization of the notion of cohomology as follows. First, we will introduce two functors denoted as K^2 and I^{-2}, complexifying the sequence C^* as in (11.1) to obtain complexes $\mathrm{K}^2 C^*$ and $\mathrm{I}^{-2} C^*$.

2.11.1 Definitions of Complexifying Functors

Let $\mathsf{Se}(\mathscr{A})$ be the category of sequences C^* as in (11.1). A morphism in f^* from C^* to ${}'C^*$ in $\mathsf{Se}(\mathscr{A})$ is a sequence of morphisms $f^j : C^j \to {}'C^j$ in $\mathscr{A}$ so that the diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & C^j & \xrightarrow{\mathrm{d}^j} & C^{j+1} & \xrightarrow{\mathrm{d}^{j+1}} & \cdots \\ & & \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{j+1}} & & \\ \cdots & \longrightarrow & {}'C^j & \xrightarrow{{}'\mathrm{d}^j} & {}'C^{j+1} & \xrightarrow{{}'\mathrm{d}^{j+1}} & \cdots \end{array} \tag{11.2}$$

is commutative, i.e., ${}'\mathrm{d}^j \circ f^j = f^{j+1} \circ \mathrm{d}^j$ for $j \in \mathbb{Z}$, the ring of integers. We will define two functors K^2 and I^{-2} from $\mathsf{Se}(\mathscr{A})$ to the category $\mathsf{Co}(\mathscr{A})$ of (cochain) complexes as follows. For $C^* \in \mathrm{Ob}(\mathsf{Se}(\mathscr{A}))$ we define

$$\begin{cases} \mathrm{K}^2 C^* := \big(\ker(\mathrm{d}^{j+1} \circ \mathrm{d}^j)\big)_{j\in\mathbb{Z}} \\ \mathrm{I}^{-2} C^* := \big(C^* \big/ \operatorname{im}(\mathrm{d}^{j-1} \circ \mathrm{d}^{j-2})\big)_{j\in\mathbb{Z}}. \end{cases} \tag{11.3}$$

Then $(\mathrm{K}^2 C^*, \mathrm{K}^2 \mathrm{d}^*)$ and $(\mathrm{I}^{-2} C^*, \mathrm{I}^{-2} \mathrm{d}^*)$ become complexes: the induced morphisms $\mathrm{K}^2\mathrm{d}^*$ and $\mathrm{I}^{-2}\mathrm{d}^*$ are defined as

$$\mathrm{K}^2 \mathrm{d}^j(x^j) := \mathrm{d}^j|_{\ker \mathrm{d}^{j+1} \circ \mathrm{d}^j}(x^j),$$

for $x^j \in \ker(\mathrm{d}^{j+1} \circ \mathrm{d}^j)$, i.e., the restriction of d^j on the subobject

$$\ker(\mathrm{d}^{j+1} \circ \mathrm{d}^j),$$

and

$$\mathrm{I}^{-2} \mathrm{d}^j([x^j]) := [\mathrm{d}^j(x^j)],$$

where $[x^j]$ denotes the class of $x^j \in C^j$ in

$$C^j \big/ \operatorname{im}(\mathrm{d}^{j-1} \circ \mathrm{d}^{j-2}),$$

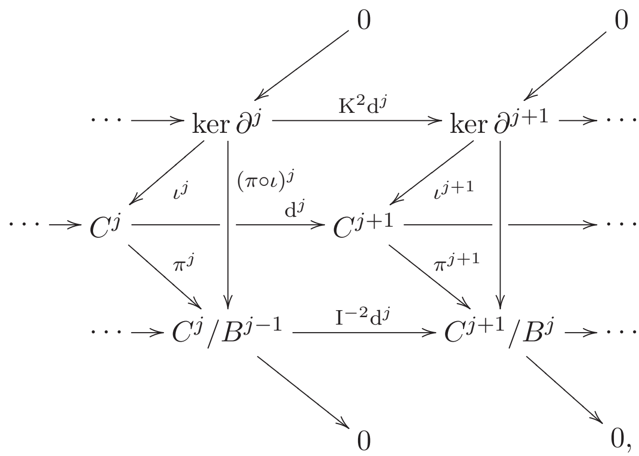
respectively. Note that

$$\begin{aligned} \ker \mathrm{I}^{-2}\mathrm{d}^j = \{[x^j] \in C^j \big/ \operatorname{im}(\mathrm{d}^{j-1} \circ \mathrm{d}^{j-2}) \mid \mathrm{d}^j(x^j - \mathrm{d}^{j-1}x^{j-1}) = 0, \\ \text{for some } x^{j-1} \in C^{j-1}\}, \end{aligned}$$

and

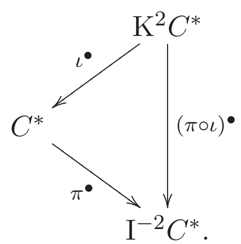
$$\begin{aligned} \operatorname{im} \mathrm{I}^{-2}\mathrm{d}^{j-1} = \{[x^j] \in C^j \big/ \operatorname{im}(\mathrm{d}^{j-1} \circ \mathrm{d}^{j-2}) \mid x^j = \mathrm{d}^{j-1}(x^{j-1}), \\ \text{for some } x^{j-1} \in C^{j-1}\}. \end{aligned}$$

We have the following diagram:



(11.4a)

where we have put $\partial^j := \mathrm{d}^{j+1} \circ \mathrm{d}^j$ and $B^j := \mathrm{im}\, \partial^{j-1}$ to simplify the diagrams. This can also be described schematically as



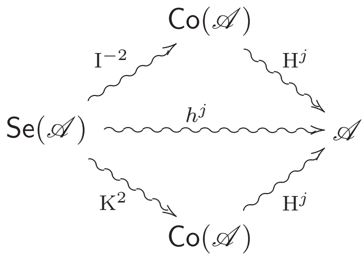
(11.4b)

2.11.2 Self-Duality of Precohomology

For C^* in $\mathsf{Se}(\mathscr{A})$ we have the two complexifying functors K^2 and I^{-2} as shown in Subsection 2.11.1. Therefore, we can consider the cohomologies of the complexes $\mathrm{K}^2 C^*$ and $\mathrm{I}^{-2} C^*$:

$$\begin{cases} \mathrm{H}^j(\mathrm{K}^2 C^*) = \ker \mathrm{K}^2 \mathrm{d}^j \big/ \mathrm{im}\, \mathrm{K}^2 \mathrm{d}^{j-1} \\ \mathrm{H}^j(\mathrm{I}^{-2} C^*) = \ker \mathrm{I}^{-2} \mathrm{d}^j \big/ \mathrm{im}\, \mathrm{I}^{-2} \mathrm{d}^{j-1}. \end{cases} \tag{11.5}$$

Then the self-duality theorem states that the morphism $(\pi \circ \iota)^\bullet$ of complexes: $\mathrm{K}^2 C^* \to \mathrm{I}^{-2} C^*$ in (11.4a) induces an isomorphism on the cohomologies in (11.5). Define $h^j C^* := \mathrm{H}^j(\mathrm{I}^{-2} C^*) \approx \mathrm{H}^j(\mathrm{K}^2 C^*)$ which is said to be the j-th *precohomology* of the sequence C^* in $\mathsf{Se}(\mathscr{A})$. We have the diagram of functors:



(11.6)

2.11.3 Proof of Self-Duality Theorem

Let ${}'h^j(C^*) := \mathrm{H}^j(\mathrm{K}^2C^*)$ and let

$$\Phi : {}'h^j(C^*) \to h^j(C^*) = \mathrm{H}^j(\mathrm{I}^{-2}C^*)$$

be the induced morphism from $(\pi \circ \iota)^\bullet : \mathrm{K}^2C^* \to \mathrm{I}^{-2}C^*$ in (11.4a). Namely, for $\bar{x} \in {}'h^j(C^*)$ we have

$$\Phi(\bar{x}) = \overline{\pi^j(\iota^j(x))}$$

where ι^j and π^j are shown in (11.4a), i.e., $\iota^j : \ker(\mathrm{d}^{j+1} \circ \mathrm{d}^j) \to C^j$ is the canonical monomorphism and $\pi^j : C^j \to C^j \big/ \mathrm{im}(\mathrm{d}^{j-1} \circ \mathrm{d}^{j-2})$ is the canonical epimorphism. Note that $x \in \ker(\mathrm{d}^{j+1} \circ \mathrm{d}^j)$ satisfies

$$\mathrm{K}^2\mathrm{d}^j(x) = \mathrm{d}^j(x) = 0$$

and that $\Phi(\bar{x}) = \overline{\pi^j(\iota^j(x))} = \overline{[x]}$ is in the j-th cohomology object $h^j(C^*) = \mathrm{H}^j(\mathrm{I}^{-2}C^*)$ where $[x] \in \ker \mathrm{I}^{-2}\mathrm{d}^j$, i.e., $\mathrm{d}^j(x - \mathrm{d}^{j-1}x^{j-1}) = 0$ for some $x^{j-1} \in C^{j-1}$ as noted in Subsection 2.11.1. We will show that Φ is a monomorphism. Let $\Phi(\bar{x}) = \overline{[x]} = 0$ in $h^j(C^*)$. As noted earlier $x = \mathrm{d}^{j-1}(x^{j-1})$ for some $x^{j-1} \in C^{j-1}$. We need to check that $x^{j-1} \in \mathrm{K}^2C^{j-1} = \ker \mathrm{d}^j \circ \mathrm{d}^{j-1}$. We compute as follows:

$$\mathrm{d}^j(\mathrm{d}^{j-1}(x^{j-1})) = \mathrm{d}^j(x) = 0.$$

Next we will show that Φ is an epimorphism. Let $\overline{[x]}$ be an arbitrary element of $h^j(C^*) = \mathrm{H}^j(\mathrm{I}^{-2}C^*)$. Since $[x]$ is in $\ker \mathrm{I}^{-2}\mathrm{d}^j$, $\mathrm{d}^j(x - \mathrm{d}^{j-1}x^{j-1}) = 0$ for some $x^{j-1} \in C^{j-1}$. Then we have $\Phi(\overline{x - \mathrm{d}^{j-1}x^{j-1}}) = \overline{[x - \mathrm{d}^{j-1}x^{j-1}]} = \overline{[x]}$ for $x - \mathrm{d}^{j-1}x^{j-1} \in \mathrm{K}^2C^j = \ker \mathrm{d}^{j+1} \circ \mathrm{d}^j$.

2.11.4 Half-Exactness of Precohomology

Let

$$0 \longrightarrow {}'C^* \xrightarrow{\alpha^*} C^* \xrightarrow{\beta^*} {}''C^* \longrightarrow 0 \tag{11.7}$$

be exact in $\mathsf{Se}(\mathscr{A})$. Then we the following exact sequence in $\mathscr{A}$:

$$h^j({}'C^*) \xrightarrow{\widetilde{\alpha^j}} h^j(C^*) \xrightarrow{\widetilde{\beta^j}} h^j({}''C^*) \tag{11.8}$$

for $j \in \mathbb{Z}$ where $\widetilde{\alpha^j} := \mathrm{H}^j(\mathrm{I}^{-2}\alpha^*)$ and $\widetilde{\beta^j} := \mathrm{H}^j(\mathrm{I}^{-2}\beta^*)$. By the self-duality of pre-cohomologies, (11.8) may also be written as

$$'h^j({}'C^*) \xrightarrow{\widehat{\alpha^j}} {}'h^j(C^*) \xrightarrow{\widehat{\beta^j}} {}'h^j({}''C^*) \tag{11.9}$$

for $j \in \mathbb{Z}$ where $\widehat{\alpha^j} := \mathrm{H}^j(\mathrm{K}^2\alpha^*)$ and $\widehat{\beta^j} := \mathrm{H}^j(\mathrm{K}^2\beta^*)$. Namely, we will prove the half-exactness of the precohomology functor from $\mathsf{Se}(\mathscr{A})$ to $\mathscr{A}$, i.e., $\ker\widetilde{\beta^j} = \operatorname{im}\widetilde{\alpha^j}$. Let $\overline{[x]} \in \ker\widetilde{\beta^j}$, i.e., $\widetilde{\beta^j}(\overline{[x]}) = \overline{[\beta^j(x)]} = 0$ in

$$h^j({}''C^*) = \ker \mathrm{I}^{-2\prime\prime}\mathrm{d}^j \big/ \operatorname{im} \mathrm{I}^{-2\prime\prime}\mathrm{d}^{j-1}.$$

Namely, $[\beta^j(x)] \in \operatorname{im}\mathrm{I}^{-2\prime\prime}\mathrm{d}^{j-1}$. As noted earlier in Subsection 2.11.1, there exists $y'' \in {}''C^{j-1}$ to satisfy $\mathrm{d}^{j-1}(y'') = \beta^j(x)$. Since β^{j-1} is epimorphic there exists $y \in C^{j-1}$ to satisfy $\beta^{j-1}(y) = y''$. Then we have

$$\begin{aligned}\beta^j(\mathrm{d}^{j-1}(y) - x) = \beta^j(\mathrm{d}^{j-1}(y)) - \beta^j(x) = {}''\mathrm{d}^{j-1}(\beta^{j-1}(y)) - \beta^j(x) = \\ = {}''\mathrm{d}^{j-1}(y'') - \beta^j(x) = 0.\end{aligned}$$

Namely, $\mathrm{d}^{j-1}(y) - x \in \ker\beta^j$. The exactness of (11.7) at C^* implies that there exists $y' \in {}'C^j$ satisfying $\alpha^j(y') = \mathrm{d}^{j-1}(y) - x$. We will prove

$$\mathrm{I}^{-2\prime}\mathrm{d}^j([y']) = 0,$$

i.e., $[y'] \in \ker \mathrm{I}^{-2\prime}\mathrm{d}^j$. By the remark on $\ker\mathrm{I}^{-2\prime}\mathrm{d}^j$ in Subsection 2.11.1 there exists $z' \in {}'C^{j-1}$ satisfying ${}'\mathrm{d}^j y' - {}'\mathrm{d}^j({}'\mathrm{d}^{j-1}(z')) = 0$. Then

$$\begin{aligned}\alpha^{j+1\prime}\mathrm{d}^j y' - \alpha^{j+1\prime}\mathrm{d}^{j\prime}\mathrm{d}^{j-1}z' &= \alpha^{j+1\prime}\mathrm{d}^j y' - \mathrm{d}^j\mathrm{d}^{j-1}\alpha^{j-1}z' \\ &= \mathrm{d}^j(\alpha^j(y') - \mathrm{d}^{j-1}\alpha^{j-1}z').\end{aligned}$$

Therefore, it is enough to show that $[\alpha^j(y')]$ is in $\ker\mathrm{I}^{-2}\mathrm{d}^j$, i.e., that we have $[\mathrm{d}^{j-1}(y) - x] \in \ker\mathrm{I}^{-2}\mathrm{d}^j$. Choose $y - x^\circ \in C^{j-1}$ where x° is chosen to satisfy $\mathrm{d}^j x - \mathrm{d}^j\mathrm{d}^{j-1}x^\circ = 0$ for $[x] \in \ker\mathrm{I}^{-2}\mathrm{d}^j$. Then

$$\begin{aligned}\mathrm{d}^j(\mathrm{d}^{j-1}y - x - \mathrm{d}^{j-1}(y - x^\circ)) = \mathrm{d}^j\mathrm{d}^{j-1}y - \mathrm{d}^j x - \mathrm{d}^j\mathrm{d}^{j-1}(y - x^\circ) = \\ = \mathrm{d}^j(\mathrm{d}^{j-1}y - x - \mathrm{d}^{j-1}(y - x^\circ)) = 0.\end{aligned}$$

Therefore, $h^j : \mathsf{Se}(\mathscr{A}) \rightsquigarrow \mathscr{A}$ is an half-exact functor.

Note 15. We will give an example of a short exact sequence of objects

$$0 \longrightarrow {}'\mathbb{Z}^* \longrightarrow \mathbb{Z}^* \longrightarrow {}''\mathbb{Z}^* \longrightarrow 0$$

in $\mathsf{Se}(\mathscr{A})$ so that the induced long sequence is not exact. That is, a precohomology sequence (h^j) is not an exact connected sequence of functors. Consider the diagram

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & & (2) \\
 & & \uparrow{\scriptstyle \text{id}} & & \uparrow{\scriptstyle \text{id}} & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & & (1) \\
 & & \uparrow & & \uparrow{\scriptstyle \text{id}} & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\text{id}} & \mathbb{Z} & \longrightarrow & 0 & & (0) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & \vdots & & \vdots & & \vdots & & & &
\end{array} \tag{11.10}$$

Complexifying (11.10) by I^{-2}, we obtain the following short exact sequence of complexes.

$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & & (2) \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 & & (1) \\
 & & \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow & & & & \\
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{\mathrm{id}} & \mathbb{Z} & \longrightarrow & 0 & & (0) \\
 & & \uparrow & & \uparrow & & \uparrow & & & & \\
 & & \vdots & & \vdots & & \vdots & & & &
\end{array}
$$

Then we get

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & h^0('\mathbb{Z}^*) & \longrightarrow & h^0(\mathbb{Z}^*) & \longrightarrow & h^0(''\mathbb{Z}^*) & \longrightarrow & h^1('\mathbb{Z}^*) \\
 & & \| & & \| & & \| & & \| \\
 & & 0 & & 0 & & \mathbb{Z} & & 0
\end{array}
$$

Namely (h^j) is not an exact connected sequence of functors.

Chapter 3

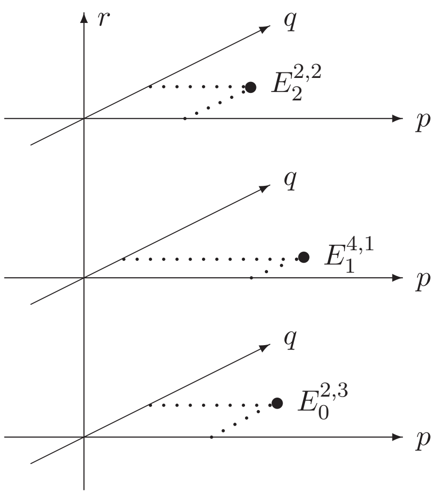
SPECTRAL SEQUENCES

3.1 Definition of Spectral Sequence

A *spectral sequence* in an abelian category $\mathscr{A}$ consists of doubly indexed objects of $\mathscr{A}$:

$$E_r^{p,q} \tag{1.1}$$

where $p, q, r \in \mathbb{Z}$. Then $E_r^{p,q}$ may be considered as an object located at the $p-$ and $q-$ axises with coordinates (p, q) at the level r. See the following figure.



There are morphisms among objects in (1.1) as follows:

$$\cdots \longrightarrow E_r^{p-r,q+r-1} \xrightarrow{\mathrm{d}_r^{p-r,q+r-1}} E_r^{p,q} \xrightarrow{\mathrm{d}_r^{p,q}} E_r^{p+r,q-r+1} \longrightarrow \cdots \tag{1.2}$$

so that the sequence of objects in (1.2) may form a complex. Namely, we can consider the cohomology at any object $E_r^{p,q}$ in (1.2). Then there is an isomorphism

$$\eta_r^{p,q} : \ker \mathrm{d}_r^{p,q} \big/ \operatorname{im} \mathrm{d}_r^{p-r,q+r-1} \xrightarrow{\approx} E_{r+1}^{p,q}. \tag{1.3}$$

That is, a sequence $\{E_r^{p,q}\}$ is said to be a *doubly indexed cohomological spectral sequence* in the abelian category $\mathscr{A}$ when condition (1.1), (1.2) and (1.3) are satisfied. If a spectral sequence begins with $E_{r_0}^{p,q}$, we sometimes write such a spectral sequence as

$$\{(E_r^{p,q}, \mathrm{d}_r^{p,q}, \eta_r^{p,q}),\ r \geq r_0,\ p,q,r,r_0 \in \mathbb{Z}\}.$$

Note 16. Let us familiarize ourselves with the behavior of a spectral sequence $E_0^{p,q}, E_1^{p,q}, \ldots$. From (1.2) we have

$$E_0^{p,q-1} = E_0^{p-0,q+0-1} \xrightarrow{\mathrm{d}_0^{p,q-1}} E_0^{p,q} \xrightarrow{\mathrm{d}_0^{p,q}} E_0^{p+0,q-0+1} = E_0^{p,q+1} .$$

Namely, in the (p,q)-coordinate the "slope" is ∞:

$$\begin{array}{c} E_0^{p,q+1} \\ \uparrow \\ E_0^{p,q} \\ \uparrow \\ E_0^{p,q-1} \end{array} \qquad\qquad E_1^{p-1,q} \xrightarrow{\mathrm{d}_1^{p-1,q}} E_1^{p,q} \xrightarrow{\mathrm{d}_1^{p,q}} E_1^{p+1,q} \tag{1.4}$$

and the "length" of $\mathrm{d}_0^{p,q}$ is 1. For $r = 1$, we have

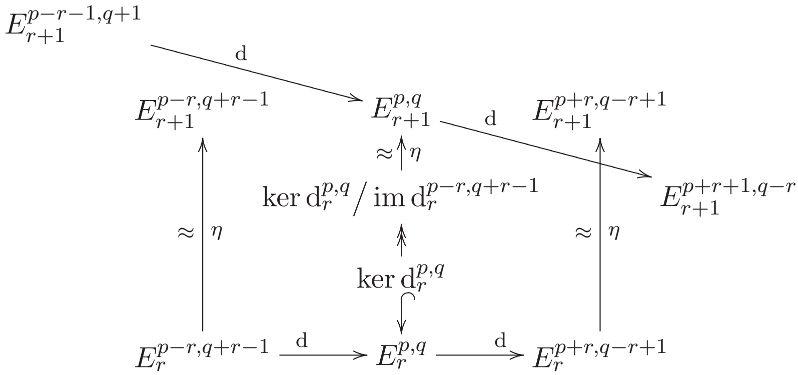
$$E_1^{p-1,q} = E_1^{p-1,q+1-1} \xrightarrow{\mathrm{d}_1^{p-1,q}} E_1^{p,q} \xrightarrow{\mathrm{d}_1^{p,q}} E_1^{p+1,q-1+1} = E_1^{p+1,q}$$

having slope 0 and length still 1. See (1.4) for $E_0^{p,q}$ and $E_1^{p,q}$. For $r = 2$, we have

$$E_2^{p-2,q+1} = E_2^{p-2,q+2-1} \xrightarrow{\mathrm{d}_2^{p-2,q+1}} E_2^{p,q} \xrightarrow{\mathrm{d}_2^{p,q}} E_2^{p+2,q-2+1} = E_2^{p+2,q-1} ,$$

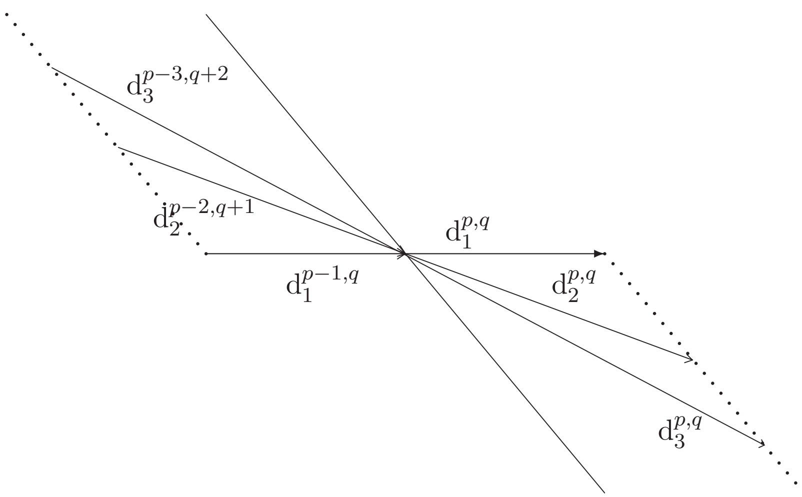
i.e., the slope is $-\frac{2-1}{2} = -\frac{1}{2}$, and the length is $\sqrt{2^2 + (2-1)^2} = \sqrt{5}$. For the general term as in (1.2), the slope of $\mathrm{d}_r^{p,q}$ in the (p,q)-coordinates is given by $\frac{-r+1}{r} = -\frac{r-1}{r}$ and the length of $\mathrm{d}_r^{p,q}$ is $\sqrt{r^2 + (r-1)^2} = \sqrt{2r^2 - 2r + 1}$.

Namely, the larger r becomes, the closer to -1 the slope becomes and the longer $\mathrm{d}_r^{p,q}$ becomes. Also notice that the isomorphism $\eta_r^{p,q}$ in (1.3) implies the following diagram:



(1.5)

namely,



For example, if $E_2^{p,q} = 0$, unless $p, q \geq 0$ (such a spectral sequence is said to be a *first quadrant spectral sequence*), then beyond $r_0 = \max(1+q, p)$, we have

$$E_{r_0+1}^{p,q} \xrightarrow{\approx} E_{r_0+2}^{p,q} \xrightarrow{\approx} \cdots \xrightarrow{\approx} E_{\infty}^{p,q} \; .$$

As explicit examples of first quadrant spectral sequences, let us consider $E_2^{2,4}$ and $E_2^{7,11}$. Our computation begins with

$$\begin{aligned} &E_2^{0,5} \xrightarrow{\mathrm{d}_2^{0,5}} E_2^{2,4} \xrightarrow{\mathrm{d}_2^{2,4}} E_2^{4,3} \\ &E_2^{5,12} \xrightarrow{\mathrm{d}_2^{5,12}} E_2^{7,11} \xrightarrow{\mathrm{d}_2^{7,11}} E_2^{9,10}. \end{aligned} \tag{1.6}$$

All terms of (1.6) are still in the first quadrant, i.e., none of them is zero (object). Therefore $E_3^{2,4}$ and $E_3^{7,11}$ are sub-quotient objects of $E_2^{2,4}$ and $E_2^{7,11}$, respectively. As for $E_3^{2,4}$ the next level $E_4^{2,4}$ is a subobject of $E_3^{2,4}$ since $\mathrm{d}_3^{-1,6} = 0$. Even though the next level $E_5^{2,4}$ is still a subobject of $E_4^{2,4}$, and the next level $E_6^{2,4}$ is still a subobject of $E_5^{2,4}$, we have

$$0 = E_6^{-4,9} \longrightarrow E_6^{2,4} \longrightarrow E_6^{8,-1} = 0.$$

Namely,

$$E_6^{2,4} \xrightarrow[\eta_6^{2,4}]{\approx} E_7^{2,4} \xrightarrow[\eta_7^{2,4}]{\approx} \cdots \xrightarrow{\approx} E_\infty^{2,4}.$$

On the other hand, for $E_2^{7,11}$, $E_9^{7,11}$ is a subobject of $E_8^{7,11}$ and then

$$E_{13}^{7,11} \xrightarrow{\approx} E_{14}^{7,11} \xrightarrow{\approx} \cdots .$$

In general, an *abutment* of a spectral sequence $(E_r^{p,q}, \mathrm{d}_r^{p,q}, \eta_r^{p,q})$, $r \geq r_0$ is a sequence

$$(E^n, \tau^{p,q})_{n,p,q\in\mathbb{Z}} \tag{1.7}$$

satisfying the following conditions (1), (2) and (3):

(1) E^n is a filtered object of $\mathscr{A}$, i.e., E^n and $\mathsf{F}_p(E^n)$ are objects of $\mathscr{A}$ such that

$$\cdots \subset \mathsf{F}_{p+1}(E^n) \subset \mathsf{F}_p(E^n) \subset \mathsf{F}_{p-1}(E^n) \subset \cdots$$

are subobjects of E^n. Then we define the p-th graded piece $\mathsf{G}_p(E^n)$ as $\mathsf{G}_p(E^n) := \mathsf{F}_p(E^n)\big/\mathsf{F}_{p+1}(E^n)$. The sequence $\mathsf{G}_\bullet(E^n) = (\mathsf{G}_p(E^n))_{p\in\mathbb{Z}}$ is said to be the *associated sequence* to the filtered object $(E^n, \mathsf{F}_\bullet(E^n))$ in $\mathscr{A}$.

(2) $E_\infty^{p,q}$ exists in $\mathscr{A}$.

(3) There is an isomorphism $\tau^{p,q} : E_\infty^{p,q} \xrightarrow{\approx} \mathsf{G}_p(E^n)$ where $n = p + q$.

When $E_\infty^{p,q}$ and $\bigoplus_{p+q=n} E_\infty^{p,q}$ are objects of the abelian category $\mathscr{A}$, we can let $E^n = \bigoplus_{p+q=n} E_\infty^{p,q}$. Then define a filtration of E^n as

$$\mathsf{F}_p(E^n) := \bigoplus_{\substack{p+q=n \\ p' \geq p}} E_\infty^{p',q}.$$

We have an isomorphism $\mathsf{G}_p(E^n) \xrightarrow{\approx} E_\infty^{p,q}$.

Remark 17. For a spectral sequence $(E_r^{p,q}, \mathrm{d}_r^{p,q}, \eta_r^{p,q})$, we will construct subobjects $Z^1(E_r^{p,q}), Z^2(E_r^{p,q}), \ldots,$ and $B^1(E_r^{p,q}), B^2(E_r^{p,q}), \ldots,$ of $E_r^{p,q}$ as follows. The subobject $Z^1(E_r^{p,q})$ consists of all those $u \in E_r^{p,q}$ satisfying $\mathrm{d}_r^{p,q}(u) = 0$ in $E_r^{p+r,q-r+1}$, and the subobject $B^1(E_r^{p,q})$ consists of all those u such that $u = \mathrm{d}_r^{p-r,q+r-1}(u')$ for some $u' \in E_r^{p-r,q+r-1}$. Then the isomorphism in (1.3)

$$\eta_r^{p,q} : Z^1(E_r^{p,q})/B^1(E_r^{p,q}) \xrightarrow{\approx} E_{r+1}^{p,q}$$

sends the class $[u]$ to $\eta_r^{p,q}([u]) \in E_{r+1}^{p,q}$ which is denoted also as $[u]$. For $[u] \in E_{r+1}^{p,q}$ to be an element of $Z^1(E_{r+1}^{p,q})$, we must have $\mathrm{d}_{r+1}^{p,q}([u]) = 0$ in $E_{r+1}^{p+r+1,q-r}$. Similarly, the isomorphism $\eta_{r+1}^{p,q}$ gives an element $[[u]] \in E_{r+2}^{p,q}$. Define a subobject $Z^2(E_r^{p,q})$ of $Z^1(E_r^{p,q})$ as follows: $u \in Z^1(E_r^{p,q})$ belongs to $Z^2(E_r^{p,q})$ if $\mathrm{d}_{r+1}^{p,q}([u]) = 0$. Define $Z^3(E_r^{p,q})$ as the subobject of $Z^1(E_r^{p,q})$ as follows: $u \in Z^1(E_r^{p,q})$ belongs to $Z^3(E_r^{p,q})$ if $\mathrm{d}_{r+1}^{p,q}([u]) = 0$ and $\mathrm{d}_{r+2}^{p,q}([[u]]) = 0$. If you let $Z^0(E_r^{p,q}) = E_r^{p,q}$, we have

$$E_r^{p,q} = Z^0(E_r^{p,q}) \supset Z^1(E_r^{p,q}) \supset Z^2(E_r^{p,q}) \supset \cdots. \tag{1.8}$$

On the other hand, let $B^0(E_r^{p,q}) = \{0\}$ and let $B^1(E_r^{p,q}) = \operatorname{im} \mathrm{d}_r^{p-r,q+r-1}$. Then define the subobject $B^2(E_r^{p,q})$ of $E_r^{p,q}$ as follows: $u \in E_r^{p,q}$ belongs to $B^2(E_r^{p,q})$ if $[u]$ in $E_{r+1}^{p,q}$ belongs to $\operatorname{im} \mathrm{d}_{r+1}^{p-r-1,q+r}$. Similarly, $u \in E_r^{p,q}$ belongs to $B^3(E_r^{p,q})$ if $[[u]] \in E_{r+2}^{p,q}$ belongs to $\operatorname{im} \mathrm{d}_{r+2}^{p-r-2,q+r+1}$. Then we have

$$\cdots \supset B^2(E_r^{p,q}) \supset B^1(E_r^{p,q}) \supset B^0(E_r^{p,q}) = \{0\}. \tag{1.9}$$

Combining (1.8) and (1.9), we obtain

$$E_r^{p,q} = Z^0(E_r^{p,q}) \supset Z^1(E_r^{p,q}) \supset \cdots \supset B^2(E_r^{p,q}) \supset B^1(E_r^{p,q}) \supset \{0\}. \tag{1.10}$$

Then we have the isomorphism

$$Z^s(E_r^{p,q})/B^s(E_r^{p,q}) \xrightarrow{\approx} E_{r+s}^{p,q}. \tag{1.11}$$

Let $Z^\infty(E_r^{p,q}) := \bigcap_s Z^s(E_r^{p,q})$ and $B^\infty(E_r^{p,q}) := \bigcup_s B^s(E_r^{p,q})$. Then we have

$$E_\infty^{p,q} \approx Z^\infty(E_r^{p,q})/B^\infty(E_r^{p,q}), \tag{1.12}$$

independent of r, i.e., for $i \geq 0$ there is an isomorphism

$$Z^\infty(E_r^{p,q})\big/B^\infty(E_r^{p,q}) \approx Z^\infty(E_{r+i}^{p,q})\big/B^\infty(E_{r+i}^{p,q}). \tag{1.13}$$

These isomorphisms in (1.11), (1.12) and (1.13) can be proved inductively from the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^2(E_r^{p,q}) & \longrightarrow & Z^2(E_r^{p,q}) & \longrightarrow & Z^2(E_r^{p,q})\big/B^2(E_r^{p,q}) & \longrightarrow & 0 \\ & & \approx\Big\downarrow\phi & & \approx\Big\downarrow\psi & & \approx\Big\downarrow\bar{\psi} & & \\ 0 & \longrightarrow & B^1(E_{r+1}^{p,q}) & \longrightarrow & Z^1(E_{r+1}^{p,q}) & \longrightarrow & E_{r+2}^{p,q} & \longrightarrow & 0 \end{array} \tag{1.14}$$

where the isomorphism $\bar{\psi}$ is induced by the isomorphisms ϕ and ψ.

3.2 Filtered Complexes

Let $C^\bullet$ be a complex in an abelian category $\mathscr{A}$, i.e., an object of $\mathsf{Co}(\mathscr{A})$. A filtration on $C^\bullet$ is defined as follows. For all p and j in $\mathbb{Z}$, $\mathsf{F}_p(C^j)$ is a subobject of C^j satisfying $\mathsf{F}_p(C^j) \supset \mathsf{F}_{p+1}(C^j)$, and $\mathrm{d}^j : C^j \to C^{j+1}$ also satisfying

$$\mathrm{d}^j|_{\mathsf{F}_p(C^j)}(\mathsf{F}_p(C^j)) \subset \mathsf{F}_p(C^{j+1})$$

for all $j, p \in \mathbb{Z}$. Then the subcomplexes $\{\mathsf{F}_p(C^\bullet)\}_{p\in\mathbb{Z}}$ satisfy

$$\mathsf{F}_p(C^\bullet) \supset \mathsf{F}_{p+1}(C^\bullet).$$

A complex $C^\bullet$ with such a filtration is said to be a *filtered complex*. Then the short exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathsf{F}_{p+1}(C^\bullet) & \xrightarrow{\iota} & \mathsf{F}_p(C^\bullet) & \xrightarrow{\pi} & \mathsf{F}_p(C^\bullet)\big/\mathsf{F}_{p+1}(C^\bullet) & \longrightarrow & 0 \\ & & & & & & \Big\| & & \\ & & & & & & \mathsf{G}_p(C^\bullet) & & \end{array} \tag{2.1}$$

induces the long exact sequence

$$\begin{array}{l} \cdots \longrightarrow \mathrm{H}^j(\mathsf{F}_{p+1}(C^\bullet)) \longrightarrow \mathrm{H}^j(\mathsf{F}_p(C^\bullet)) \longrightarrow \\ \longrightarrow \mathrm{H}^j(\mathsf{G}_p(C^\bullet)) \longrightarrow \mathrm{H}^{j+1}(\mathsf{F}_{p+1}(C^\bullet)) \longrightarrow \cdots \end{array} \tag{2.2}$$

Then define

$$\begin{cases} V^{p,j-p} := \mathrm{H}^j(\mathsf{F}_p(C^\bullet)) \\ E^{p,j-p} := \mathrm{H}^j(\mathsf{G}_p(C^\bullet)). \end{cases} \tag{2.3}$$

We can re-write the long exact sequence (2.2) as

$$\begin{array}{ccc} (V^{p,j-p}) & \xrightarrow{\quad t^{p,j-p} \quad} & (V^{p,j-p}) \\ & {\scriptstyle k^{p,j-p}}\nwarrow \qquad \swarrow{\scriptstyle h^{p,j-p}} & \\ & (E^{p,j-p}) & \end{array} \tag{2.4}$$

where

$$t^{p,j-p} := \mathrm{H}^j(\mathsf{F}_p(C^\bullet) \overset{\iota}{\hookrightarrow} \mathsf{F}_{p-1}(C^\bullet)) \ ,$$

and

$$h^{p,j-p} := \mathrm{H}^j(\mathsf{F}_p(C^\bullet) \xrightarrow{\ \pi\ } \mathsf{G}_p(C^\bullet))$$

and $k^{p,j-p}$ is the connecting morphism ∂^j in Chapter II. Note also that the bi-degrees of $t^{p,j-p}$, $h^{p,j-p}$ and $k^{p,j-p}$ are $(-1,+1)$, $(0,0)$ and $(+1,0)$ respectively. Namely, a filtered complex $(C^\bullet, (\mathsf{F}_p(C^\bullet))_{p\in\mathbb{Z}})$ induces a spectral sequence beginning $E_1^{p,q}$, $p+q=j$. When $E_r^{p,q}$ is the initial term, the bi-degree of $k^{p,q}$ becomes $(r,-r+1)$. Then the composition $h^{p,q} \circ k^{p,q}$ is $\mathrm{d}^{p,q} : E_r^{p,q} \to E_r^{p+r,q-e+1}$. (The long exact sequence in (2.4) induced by a filtered complex is an example of an *exact couple*. See Lubkin, *Cohomology of completions* [LuCo].)

3.3 Double Complexes

Let $\mathscr{A}$ be an abelian category. For $(p,q) \in \mathbb{Z}\times\mathbb{Z}$, let $C^{p,q}$ be an object of $\mathscr{A}$ and let $\mathrm{d}_{(1,0)}^{p,q} : C^{p,q} \to C^{p+1,q}$ and $\mathrm{d}_{(0,1)}^{p,q} : C^{p,q} \to C^{p,q+1}$ be morphisms satisfying $\mathrm{d}_{(1,0)}^{p+1,q} \circ \mathrm{d}_{(1,0)}^{p,q} = 0$ and $\mathrm{d}_{(0,1)}^{p,q+1} \circ \mathrm{d}_{(0,1)}^{p,q} = 0$. Namely, $(C^{\bullet,q}, \mathrm{d}_{(1,0)}^{\bullet,q})$ and $(C^{p,\bullet}, \mathrm{d}_{(0,1)}^{p,\bullet})$ are complexes. In the following diagram:

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & C^{p,q+1} & \xrightarrow{\ \mathrm{d}_{(1,0)}^{p+1,q}\ } & C^{p+1,q+1} & \longrightarrow & \cdots \\ & & {\scriptstyle \mathrm{d}_{(0,1)}^{p,q}}\uparrow & & \uparrow{\scriptstyle \mathrm{d}_{(0,1)}^{p+1,q}} & & \\ \cdots & \longrightarrow & C^{p,q} & \xrightarrow{\ \mathrm{d}_{(1,0)}^{p,q}\ } & C^{p+1,q} & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array} \tag{3.1}$$

we also let $\mathrm{d}^{p+1,q}_{(0,1)} \circ \mathrm{d}^{p,q}_{(1,0)} = \mathrm{d}^{p+1,q}_{(1,0)} \circ \mathrm{d}^{p,q}_{(0,1)}$ be satisfied. Then

$$(C^{p,q}, \mathrm{d}^{p,q}_{(1,0)}, \mathrm{d}^{p,q}_{(0,1)})_{p,q\in\mathbb{Z}}$$

is said to be a *double complex* in $\mathscr{A}$. Next, for a double complex we associate a complex as follows. Let

$$C^n = \bigoplus_{p+q=n} C^{p,q}. \tag{3.2}$$

For the inclusion $\iota^{p,q} : C^{p,q} \hookrightarrow C^n$, define $\mathrm{d}^n : C^n \to C^{n+1}$ as

$$\mathrm{d}^n|_{C^{p,q}} = \iota^{p+1,q} \circ \mathrm{d}^{p,q}_{(1,0)} + (-1)^n \iota^{p,q+1} \circ \mathrm{d}^{p,q}_{(0,1)}. \tag{3.3}$$

Then we have $\mathrm{d}^{n+1} \circ \mathrm{d}^n = 0$ where d^n is understood as the sum of those morphisms in (3.3). Namely, we have obtained the complex $(C^\bullet, \mathrm{d}^\bullet)$ associated with the double complex $(C^{\bullet,\bullet}, \mathrm{d}^{\bullet,\bullet}_{(1,0)}, \mathrm{d}^{\bullet,\bullet}_{(0,1)})$. Next we will define an appropriate filtration on the associated complex $(C^\bullet, \mathrm{d}^\bullet)$ so that we may have a filtered complex $(C^\bullet, (\mathsf{F}_p(C^\bullet))_{p\in\mathbb{Z}})$. Define

$$\mathsf{F}_p(C^n) := \bigoplus_{p'\geq p} C^{p',n-p'} = \bigoplus_{\substack{p'+q=n\\ p'\geq p}} C^{p',q} \tag{3.4}$$

for $p, q, n \in \mathbb{Z}$. Then $\mathsf{F}_p(C^n)$ is a subobject of C^n satisfying

$$\mathsf{F}_p(C^n) \supset \mathsf{F}_{p+1}(C^n)$$

and also $\mathrm{d}^n : C^n \to C^{n+1}$ satisfies $\mathrm{d}^n|_{\mathsf{F}_p(C^n)}(\mathsf{F}_p(C^n)) \subset \mathsf{F}_p(C^{n+1})$. Namely, we have obtained the filtered complex associated with the double complex. As before, we have the short exact sequence

$$0 \longrightarrow \mathsf{F}_{p+1}(C^n) \longrightarrow \mathsf{F}_p(C^n) \longrightarrow \mathsf{G}_p(C^n) \longrightarrow 0\ .$$

Notice that

$$\mathsf{G}_p(C^n) = \mathsf{F}_p(C^n)/\mathsf{F}_{p+1}(C^n) = \bigoplus_{p'\geq p} C^{p',n-p'} \Big/ \bigoplus_{p'\geq p+1} C^{p',n-p'} \approx C^{p,n-p}.$$

That is, $\mathrm{d}^n : C^n \to C^{n+1}$ induces $\mathrm{d}^n : \mathsf{F}_p(C^n) \to \mathsf{F}_{p+1}(C^n)$ which induces $\mathsf{G}_p(C^n) \approx C^{p,n-p} \to C^{p,n+1-p} \approx \mathsf{G}_p(C^{n+1})$. Namely, we get

$$\begin{array}{ccccccc} C^{n+1} & \supset & C^{p,n+1-p} & = & C^{p,q+1} & \approx & \mathsf{G}_p(C^{n+1}) \\ {\scriptstyle \mathrm{d}^n}\uparrow & & \uparrow{\scriptstyle \mathrm{d}^{p,n-p}_{(0,1)}} & & {\scriptstyle \mathrm{d}^{p,q}_{(0,1)}}\uparrow & & \uparrow \\ C^n & \supset & C^{p,n-p} & = & C^{p,q} & \approx & \mathsf{G}_p(C^n) \end{array} \tag{3.5}$$

From the exact couple in (2.2) and (2.4), we have

$$E_1^{p,q} = \mathrm{H}^n(\mathsf{G}_p(C^\bullet)) \qquad \text{and} \qquad V^{p,n-p} = \mathrm{H}^n(\bigoplus_{p' \geq p} C^{p',\bullet - p'}) \tag{3.6}$$

where

$$\mathrm{H}^n(\mathsf{G}_p(C^\bullet)) = \mathrm{H}_\uparrow^{n-p}(C^{p,\bullet}) = \mathrm{H}_\uparrow^q(C^{p,\bullet}) = \ker \mathrm{d}_{(0,1)}^{p,q} \big/ \operatorname{im} \mathrm{d}_{(0,1)}^{p,q-1}.$$

Namely, we have the spectral sequence of slope zero and of length 1 at level 1:

$$\begin{array}{l} E_1^{0,1} = \mathrm{H}_\uparrow^1(C^{0,\bullet}) \xrightarrow{\mathrm{d}_1^{0,1}} E_1^{1,1} = \mathrm{H}_\uparrow^1(C^{1,\bullet}) \longrightarrow \cdots \\ \\ E_1^{0,0} = \mathrm{H}_\uparrow^0(C^{0,\bullet}) \longrightarrow E_1^{1,0} = \mathrm{H}_\uparrow^0(C^{1,\bullet}) \longrightarrow \cdots . \end{array} \tag{3.7}$$

Since $E_1^{p,q}$ is obtained by taking cohomologies in the direction of the q-axis (i.e., vertically), we may begin at level zero. Namely, the initial term begins

$$E_0^{p,q} = C^{p,q} \tag{3.8}$$

whose level zero is expressed as:

$$\begin{array}{ccc} \vdots & & \vdots \\ \uparrow & & \uparrow \\ E_0^{0,1} = C^{0,1} & & E_0^{1,1} = C^{1,1} \\ \uparrow {\scriptstyle \mathrm{d}_0^{0,0} = \mathrm{d}_{(0,1)}^{0,0}} & & \uparrow {\scriptstyle \mathrm{d}_0^{1,0} = \mathrm{d}_{(0,1)}^{1,0}} \\ E_0^{0,0} = C^{0,0} & & E_0^{1,0} = C^{1,0} \end{array}$$

That is, $\mathrm{H}_\uparrow^q(E_0^{p,\bullet}) \approx E_1^{p,q}$ holds. Furthermore, we have $E_2^{p,q} = \mathrm{H}_\rightarrow^p(E_1^{\bullet,q})$. Consequently, we obtain

$$E_2^{p,q} = \mathrm{H}_\rightarrow^p(E_1^{\bullet,q}) = \mathrm{H}_\rightarrow^p(\mathrm{H}_\uparrow^q(C^{\bullet,\bullet})). \tag{3.9}$$

3.3.1 Abutment of Double Complex Spectral Sequence

For the filtration of C^n defined in (3.4), let us assume $\mathsf{F}_p(C^n) = 0$ if p is greater than a certain p_0 depending upon n, and $\mathsf{F}_p(C^n) = C^n$ if p is less than a certain p_0' also depending upon n. Note that for a double complex in the first quadrant, i.e., $C^{p,q} = 0$ unless $p, q \geq 0$, the above conditions are satisfied. For the exact sequence

$$0 \longrightarrow \mathsf{F}_p(C^n) \longrightarrow C^n \longrightarrow C^n \big/ \mathsf{F}_p(C^n) \longrightarrow 0, \tag{3.10}$$

the j-th cohomology $V^{p,j-p} = \mathrm{H}^j(\mathsf{F}_p(C^\bullet)) = 0$ for $p > p_0$ in (2.3). Then in (2.4), the morphism $t^{p,j-p} = \mathrm{H}^j(\mathsf{F}_p(C^\bullet) \hookrightarrow \mathsf{F}_{p-1}(C^\bullet))$ becomes an isomorphism. The exact sequence (2.2) implies that $E^{p,j-p} = \mathrm{H}^j(\mathsf{G}_p(C^\bullet)) = 0$. Namely, for any j and p, there exists r_0 such that

$$E_{r_0}^{p,j-p} \approx E_{r_0+1}^{p,j-p} \approx \cdots \approx E_\infty^{p,j-p}.$$

Also, for such a filtration on C^n we have $V^{p,j-p} = \mathrm{H}^j(\mathsf{F}_p(C^\bullet)) = 0$ as noted in the above, and $\mathrm{H}^j(\mathsf{F}_p(C^\bullet)) \approx \mathrm{H}^j(C^\bullet)$ for a small p. Then the induced filtration on $\mathrm{H}^j(C^\bullet)$ also satisfies the finiteness conditions of the filtration of C^n in the above.

For the long exact sequence (2.4), we can derive another long exact sequence:

$$\begin{array}{ccc} (V_r^{p,j-p}) & \xrightarrow{(t_r^{p,j-p})} & (V_r^{p,j-p}) \\ {\scriptstyle (k_r^{p,j-p})}\nwarrow & & \swarrow{\scriptstyle (h_r^{p,j-p})} \\ & (E_r^{p,j-p}) & \end{array} \tag{3.11}$$

where $V_1^{p,j-p} = \operatorname{im} t^{p,j-p}$ and

$$E_1^{p,j-p} = \ker(h^{p,j-p} \circ k^{p,j-p}) / \operatorname{im}(h^{p-1,j-p} \circ k^{p-1,j-p}),$$

i.e., the cohomology of

$$E^{p-1,j-p} \xrightarrow{h^{p-1,j-p} \circ k^{p-1,j-p}} E^{p,j-p} \xrightarrow{h^{p,j-p} \circ k^{p,j-p}} E^{p+1,j-p}$$

and the higher $V_r^{p,j-p}$ and $E_r^{p,j-p}$ can be defined inductively. Namely,

$$E_r = k^{-1}(\operatorname{im} t^r) / h(t^{-r}(0))$$

and $E_r \to E_r$ is the induced morphism by $h \circ t^{-r} \circ k$ where the double indices are omitted. Therefore, the long exact sequence (3.11) becomes

$$\begin{array}{ccc} V_{r_0}^{p+1,j-p-1} & \xrightarrow{t_{r_0}^{p+1,j-p+1}} & V_{r_0}^{p,j-p} \\ V_{r_0}^{p+r_0,j-p-r_0+1} & & \\ {\scriptstyle k_{r_0}^{p,j-p}}\nwarrow & & \swarrow{\scriptstyle h_{r_0}^{p,j-p}} \\ & E_{r_0}^{p,j-p} \approx E_\infty^{p,j-p} & \end{array} \tag{3.12}$$

where we have $V_{r_0}^{p+1,j-p-1} = \mathsf{F}_{p+1}(\mathrm{H}^j(C^\bullet))$, $V_{r_0}^{p,j-p} = \mathsf{F}_p(\mathrm{H}^j(C^\bullet))$ and $V_{r_0}^{p+r_0,j-p-r_0+1} = \mathsf{F}_{p+r_0}(\mathrm{H}^{j+1}(C^\bullet))$. As mentioned above the induced filtration on $\mathrm{H}^j(C^\bullet)$ also satisfies the finiteness conditions. We can find a large r_0 so that we may have

$$E_{r_0}^{p,j-p} \approx E_\infty^{p,j-p} \approx \mathsf{G}_p(\mathrm{H}^j(C^\bullet)) = \mathsf{F}_p(\mathrm{H}^j(C^\bullet))\big/\mathsf{F}_{p+1}(\mathrm{H}^j(C^\bullet)).$$

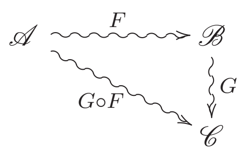
Therefore, $\mathrm{H}^j(C^\bullet)$ is an abutment of the spectral sequence of a double complex $(C^{\bullet,\bullet}, \mathrm{d}^{\bullet,\bullet})$.

3.3.2 Composite Functors

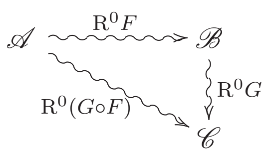
Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be abelian categories. We also assume that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. Let $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ and $G : \mathscr{B} \rightsquigarrow \mathscr{C}$ be left exact additive functors. Furthermore, assume that for every injective object I of $\mathscr{A}$ we have

$$\mathrm{R}^j G(FI) = 0, \qquad \text{for } j > 0. \tag{3.13}$$

Since we have (D.F.0) in Section 2.8, the diagram



induces the commutative diagram



i.e., $\mathrm{R}^0G \circ \mathrm{R}^0F \approx G \circ F \approx \mathrm{R}^0(G \circ F)$. Note that this commutativity will play an important role for the notion of a derived category in Chapter IV.

3.3.3 Cartan–Eilenberg Resolution

For an arbitrary object A of $\mathscr{A}$, let $(I^\bullet, \mathrm{d}^\bullet)$ be an injective resolution of A. For the functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ in Subsection 3.3.2, $FI^\bullet$ is a complex in $\mathscr{B}$. Then an injective resolution $Q^{\bullet,\bullet}$ of the complex $FI^\bullet$ is said to be a *Cartan–Eilenberg resolution* of $FI^\bullet$. That is, for $p, q \in \mathbb{Z}$, $Q^{p,q}$ is an injective object so that $(Q^{p,q})_{p,q\in\mathbb{Z}}$ forms a double complex in $\mathscr{B}$, and furthermore $Q^{p,\bullet}$ is an injective resolution of FI^p. Namely, in the following diagram in the first quadrant

$$\begin{array}{ccccccccc}
 & \vdots & & \vdots & & \vdots & & \vdots & \\
 & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & 0 & \longrightarrow & Q^{0,1} & \longrightarrow & Q^{1,1} & \longrightarrow & Q^{2,1} & \longrightarrow \cdots \\
 & \uparrow & & \uparrow {\scriptstyle \mathrm{d}^{0,0}_{(0,1)}} & & \uparrow {\scriptstyle \mathrm{d}^{1,0}_{(0,1)}} & & \uparrow {\scriptstyle \mathrm{d}^{2,0}_{(0,1)}} & \\
\cdots \longrightarrow & 0 & \longrightarrow & Q^{0,0} & \xrightarrow{\mathrm{d}^{0,0}_{(1,0)}} & Q^{1,0} & \xrightarrow{\mathrm{d}^{1,0}_{(1,0)}} & Q^{2,0} & \longrightarrow \cdots \\
 & \uparrow & & \uparrow {\scriptstyle \epsilon^0} & & \uparrow {\scriptstyle \epsilon^1} & & \uparrow {\scriptstyle \epsilon^2} & \\
\cdots \longrightarrow & 0 & \longrightarrow & FI^0 & \xrightarrow{F\mathrm{d}^0} & FI^1 & \xrightarrow{F\mathrm{d}^1} & FI^2 & \longrightarrow \cdots
\end{array} \tag{3.14}$$

each vertical sequence is the injective resolution of FI^p. We will prove that such a resolution $Q^{\bullet,\bullet}$ of $FI^\bullet$ exists. First, decompose the complex $FI^\bullet$ as follows:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & \ker F\mathrm{d}^j & & \ker F\mathrm{d}^{j+1} & & \\
 & & \downarrow {\scriptstyle \iota^j} & \nearrow & \downarrow {\scriptstyle \iota^{j+1}} & & \\
\cdots \longrightarrow & & FI^j & \xrightarrow{F\mathrm{d}^j} & FI^{j+1} & \longrightarrow \cdots & \\
 & & \downarrow {\scriptstyle \rho^j} & \nearrow {\scriptstyle \alpha^{j+1}} & \downarrow {\scriptstyle \rho^{j+1}} & & \\
 & & \operatorname{im} F\mathrm{d}^j & & \operatorname{im} F\mathrm{d}^{j+1} & & \\
 & \nearrow & \downarrow & & \downarrow & & \\
 & 0 & 0 & & 0 & &
\end{array} \tag{3.15}$$

From (3.15) we extract the short exact sequences

$$0 \longrightarrow \ker F\mathrm{d}^j \xrightarrow{\iota^j} FI^j \xrightarrow{\rho^j} \operatorname{im} F\mathrm{d}^j \longrightarrow 0$$

$$0 \longrightarrow \operatorname{im} F\mathrm{d}^{j-1} \xrightarrow{\alpha^j} \ker F\mathrm{d}^j \xrightarrow{\pi^j} \ker F\mathrm{d}^j \big/ \operatorname{im} F\mathrm{d}^{j-1} \longrightarrow 0 \tag{3.16}$$

$$\ker F\mathrm{d}^j \big/ \operatorname{im} F\mathrm{d}^{j-1} = \mathrm{H}^j(FI^\bullet)$$

where π^j is the canonical epimorphism. For the objects $\operatorname{im} F\mathrm{d}^{j-1}$ and $\mathrm{H}^j(FI^\bullet)$ of the abelian category $\mathscr{B}$ with enough injectives, let $'\mathcal{I}^\bullet$ and $''\mathcal{I}^\bullet$ be injective resolutions of $\operatorname{im} F\mathrm{d}^{j-1}$ and $\mathrm{H}^j(FI^\bullet)$ respectively. Then in the second exact sequence of (3.14) the direct product of $'\mathcal{J}^\bullet$ and $''\mathcal{J}^\bullet$ becomes an acyclic complex consisting of injective objects. Consequently, we obtain an injective resolution of $\ker F\mathrm{d}^j$. From the first exact sequence in (3.16) we similarly obtain an injective resolution of FI^j which we denote as $Q^{j,\bullet}$. Note that we have

$$F\mathrm{d}^j = \iota^{j+1} \circ \alpha^{j+1} \circ \rho^j$$

in (3.15). From the above construction of $Q^{j,\bullet}$, for $j = 0, 1, 2, \ldots$, we obtain the double complex $Q^{\bullet,\bullet}$ consisting of injective objects of $\mathscr{B}$ so that in the following diagram:

$$\begin{array}{ccccccccc} \cdots \longrightarrow & Q^{j-1,\bullet} & \xrightarrow{\mathrm{d}^{j-1,\bullet}_{(1,0)}} & Q^{j,\bullet} & \xrightarrow{\mathrm{d}^{j,\bullet}_{(1,0)}} & Q^{j+1,\bullet} & \longrightarrow \cdots \\ & \uparrow{\scriptstyle \epsilon^{j-1}} & & \uparrow{\scriptstyle \epsilon^{j}} & & \uparrow{\scriptstyle \epsilon^{j+1}} & \\ \cdots \longrightarrow & FI^{j-1} & \xrightarrow{F\mathrm{d}^{j-1}} & FI^j & \xrightarrow{F\mathrm{d}^{j}} & FI^{j+1} & \longrightarrow \cdots \end{array} \tag{3.17}$$

$(Q^{j,\bullet}, \mathrm{d}^{j,\bullet}_{(0,1)})$ is an injective resolution of FI^j. Furthermore, $\ker \mathrm{d}^{j,\bullet}_{(1,0)}$, $\operatorname{im} \mathrm{d}^{j-1,\bullet}_{(1,0)}$ and $\ker \mathrm{d}^{j,\bullet}_{(1,0)} \big/ \operatorname{im} \mathrm{d}^{j-1,\bullet}_{(1,0)}$ are injective resolutions of the objects $\ker \mathrm{d}^j$, $\operatorname{im} \mathrm{d}^{j-1}$ and $\mathrm{H}^j(FI^\bullet)$, respectively.

3.3.4 Spectral Sequence of Composite Functor

For the double complex $Q^{\bullet,\bullet}$ in the first quadrant, $G : \mathscr{B} \rightsquigarrow \mathscr{C}$ gives the double complex $GQ^{\bullet,\bullet}$ in $\mathscr{C}$ as follows:

$$\begin{array}{ccccccc} \vdots & & \vdots & & \vdots & & \\ \uparrow & & \uparrow & & \uparrow & & \\ GQ^{0,2} & \longrightarrow & GQ^{1,2} & \longrightarrow & GQ^{2,2} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ GQ^{0,1} & \longrightarrow & GQ^{1,1} & \longrightarrow & GQ^{2,1} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & \uparrow & & \\ GQ^{0,0} & \longrightarrow & GQ^{1,0} & \longrightarrow & GQ^{2,0} & \longrightarrow & \cdots \end{array} \tag{3.18}$$

Since $0 \to FI^p \xrightarrow{\epsilon^p} Q^{p,\bullet}$ is an injective resolution as noted, the hypothesis $\mathrm{R}^q G(FI^p) = 0$ in (3.13) implies $\mathrm{H}^q(GQ^{p,\bullet}) = 0$ for $q = 1, 2, \ldots$. That is, the vertical sequences in (3.18) are exact. Namely, cohomologies in the q-axis direction are all zero for $q \geq 1$. Using the notation in Section 3.3, we have

$\mathrm{H}^q_{\uparrow}(CQ^{p,\bullet}) = 0$ for $q \geq 1$. Therefore, $E_1^{p,q}$ can be computed as follows:

$$E_1^{p,q} = \mathrm{H}^q_{\uparrow}(GQ^{p,\bullet}) = \begin{cases} 0, \text{ for } q \geq 1 \\ \mathrm{H}^0_{\uparrow}(GQ^{p,\bullet}) = \mathrm{R}^0G(FI^p) \approx G(FI^p), \\ \quad \text{for } q = 0. \end{cases} \tag{3.19}$$

Let us draw the spectral sequence (3.19) at level 1 of slope 0 as follows:

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & 0 & \xrightarrow{\mathrm{d}_1^{0,1}} & 0 & \xrightarrow{\mathrm{d}_1^{1,1}} & 0 & \xrightarrow{\mathrm{d}_1^{2,1}} & \cdots \\ \\ \cdots & \longrightarrow & 0 & \longrightarrow & E_1^{0,0} & \xrightarrow{\mathrm{d}_1^{0,0}} & E_1^{1,0} & \xrightarrow{\mathrm{d}_1^{1,0}} & E_1^{2,0} & \longrightarrow & \cdots \\ & & & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & G(FI^0) & \xrightarrow{\mathrm{d}_1^{0,0}} & G(FI^1) & \xrightarrow{\mathrm{d}_1^{1,0}} & G(FI^2) & \longrightarrow & \cdots \end{array} \tag{3.20}$$

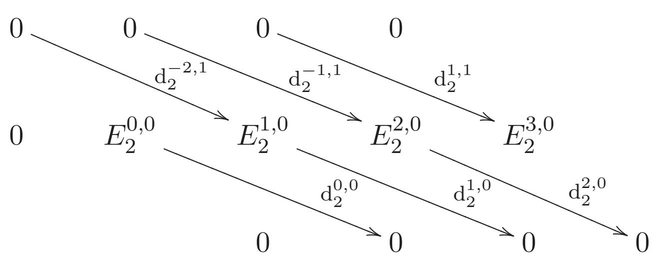
Then the $E_2^{p,0}$ terms are the cohomologies of the complex

$$(E_1^{\bullet,0}, \mathrm{d}_1^{\bullet,0}) = (G(FI^\bullet), G(F\mathrm{d}^\bullet)).$$

Namely, we have

$$\begin{aligned} E_2^{p,0} &= \mathrm{H}^p_{\rightarrow}(E_1^{\bullet,0}) = \\ &= \mathrm{H}^p_{\rightarrow}(\mathrm{H}^0_{\uparrow}(GQ^{\bullet,\bullet})) = \\ &= \mathrm{H}^p_{\rightarrow}(\mathrm{R}^0G(FI^\bullet)) \approx \\ &\approx \mathrm{H}^p_{\rightarrow}((G \circ F)I^\bullet) = \\ &= \ker \mathrm{d}_1^{p,0} \big/ \operatorname{im} \mathrm{d}_1^{p-1,0} = \\ &= \ker(G \circ F)\mathrm{d}^p \big/ \operatorname{im}(G \circ F)\mathrm{d}^{p-1} = \\ &= \mathrm{H}^p_{\rightarrow}((G \circ F)I^\bullet) = \\ &= \mathrm{R}^p(G \circ F)A. \end{aligned} \tag{3.21}$$

The spectral sequence (3.21) at level 2 with slope $-\frac{1}{2}$ has non-vanishing terms on the p-axis as follows:



(3.22)

Consequently, we get $E_2^{p,0} \approx E_3^{p,0} \approx \cdots \approx E_\infty^{p,0}$. Then the spectral sequence associated with the double complex (3.18) abuts upon $E^{p+q} = E^n$, the total cohomology $\mathrm{H}^n(\bigoplus_{p+q=\bullet} GQ^{p,q})$. From (3.20) the terms with $q = 0$ are the only non-vanishing terms. Namely, we have

$$E^p = \mathrm{H}^p(\bigoplus_{p+0=\bullet} GQ^{p,0}) = \mathrm{R}^p(G \circ F)A.$$

Since $\tau^{p,0} : G^p(E^p) \approx E_\infty^{p,0}$, where $E^p := \bigoplus_{p+q=p} E_\infty^{p,q} \approx E_\infty^{p,0}$, we have $E_\infty^{p,0} \approx E_2^{p,0} = \mathrm{R}^p(G \circ F)A$ as the abutment E^p.

For the double complex (3.18), $C^{p,q} := GQ^{p,q}$, define another filtration on $C^n = \bigoplus_{p+q=n} C^{p,q} = \bigoplus_{p+q=n} GQ^{p,q}$ as follows:

$$'\mathsf{F}^p C^n := \bigoplus_{\substack{q+p'=n \\ p' \geq p}} C^{q,p'} = \bigoplus_{\substack{p+p'=n \\ p' \geq p}} GQ^{q,p'}. \tag{3.23}$$

Just as for the previous filtration $\mathsf{F}^p C^n$ the following spectral sequences are induced:

$$\begin{aligned} {}'E_0^{p,q} &:= C^{q,p} = GQ^{q,p} \\ {}'E_1^{p,q} &:= \mathrm{H}^q_{\rightarrow}(GQ^{\bullet,p}) \\ {}'E_2^{p,q} &:= \mathrm{H}^p_{\uparrow}(\mathrm{H}^q_{\rightarrow}(GQ^{\bullet,\bullet})), \end{aligned} \tag{3.24}$$

abutting upon $\mathrm{R}^n(G \circ F)A$ as well. Recall that the injective resolution of the middle object of (3.16) was the direct sum of the injective resolutions of the left and right objects. Namely, the short exact sequence in (3.16) is a split exact sequence. Hence we have ${}'E_1^{p,q} = G(\mathrm{H}^q_{\rightarrow}(Q^{\bullet,p})) \overset{\approx}{\longleftarrow} \mathrm{H}^q_{\rightarrow}(GQ^{\bullet,p})$. Then ${}'E_2^{p,q}$ in (3.24) becomes

$$\begin{aligned} {}'E_2^{p,q} &= \mathrm{H}^p_{\uparrow}(\mathrm{H}^q_{\rightarrow}(GQ^{\bullet,\bullet})) \approx \\ &\approx \mathrm{H}^p_{\uparrow}(G(\mathrm{H}^q_{\rightarrow}(Q^{\bullet,\bullet}))) \approx \\ &\approx \mathrm{R}^p G(\mathrm{H}^q_{\rightarrow}(FI^\bullet)), \end{aligned} \tag{3.25}$$

where the last isomorphism holds since $\mathrm{H}^q_{\rightarrow}(Q^{\bullet,\bullet})$ is an injective resolution of $\mathrm{H}^q_{\rightarrow}(FI^\bullet)$. Furthermore, since $I^\bullet$ is an injective resolution of A, we have $\mathrm{H}^q_{\rightarrow}(FI^\bullet) = \mathrm{R}^q FA$. That is, ${}'E_2^{p,q} = \mathrm{R}^p G(\mathrm{R}^q FA)$, completing the proof of: a spectral sequence associated with a double complex implies a spectral sequence induced by a composite functor,

$$E_2^{p,q} = \mathrm{R}^p G(\mathrm{R}^q FA) \quad \text{abutting upon} \quad \mathrm{R}^n(G \circ F)A, \tag{3.26}$$

where $n = p + q$.

3.3.5 The converse

Let $A^\bullet$ be an object of the subcategory $\mathsf{Co}^+(\mathscr{A})$ of bounded from below objects of $\mathsf{Co}(\mathscr{A})$ as in Notation 12 in Chapter II. Namely, an object $A^\bullet$ of $\mathsf{Co}^+(\mathscr{A})$ is a complex satisfying $A^j = 0$ for $j < 0$. We will prove that the j-th cohomology $\mathrm{H}^j(A^\bullet)$ is the j-th derived functor of the 0-th cohomology of this complex. That is,

$$\mathrm{H}^j(A^\bullet) = \mathrm{R}^j\mathrm{H}^0_{A^\bullet}(A^\bullet) = \mathrm{R}^j \ker(A^0 \xrightarrow{\mathrm{d}^0} A^1). \tag{3.27}$$

First we will characterize an injective object of $\mathsf{Co}^+(\mathscr{A})$ as follows. An object $I^\bullet$ of $\mathsf{Co}^+(\mathscr{A})$ is injective if each I^j is an injective object of $\mathscr{A}$, and

$$\cdots \longrightarrow 0 \longrightarrow I^0 \xrightarrow{\mathrm{d}^0} I^1 \xrightarrow{\mathrm{d}^1} I^2 \xrightarrow{\mathrm{d}^2} \cdots \tag{3.28}$$

is exact, i.e., $\ker \mathrm{d}^j = \mathrm{im}\, \mathrm{d}^{j-1}$ for $j = 0, 1, 2, \ldots$, where d^{-1} is the zero morphism and $\ker \mathrm{d}^0$ is an injective object of $\mathscr{A}$. In order to prove the statement (3.27) we will use the Buchsbaum Theorem which asserts the following. For this exact connected sequence of functors $\mathrm{H}^j : \mathsf{Co}^+(\mathscr{A}) \rightsquigarrow \mathscr{A}$, H^j becomes the derived functor of $\mathrm{H}^0(A^\bullet) = \ker(A^0 \xrightarrow{\mathrm{d}^0} A^1)$ if the following condition is satisfied:

$$\begin{array}{l}\text{For an arbitrary object } A^\bullet \text{ of } \mathsf{Co}^+(\mathscr{A}), \text{ there exists a subobject } {}'A^\bullet \\ \text{of } A^\bullet \text{ satisfying } \mathrm{H}^j({}'A^\bullet) = 0, \text{ for } j \geq 1.\end{array} \tag{3.29}$$

We construct such a subobject ${}'A^\bullet$ as follows. Define ${}'A^\bullet$ as given in the diagram below:

$$
\begin{array}{ccccccc}
A^\bullet : \cdots \longrightarrow & A^{j-2} & \xrightarrow{\mathrm{d}^{j-2}} & A^{j-1} & \xrightarrow{\mathrm{d}^{j-1}} & A^j & \xrightarrow{\mathrm{d}^j} \cdots \\
& \| & & \downarrow & & & \\
{}'A^\bullet : \cdots \longrightarrow & {}'A^{j-2} & \xrightarrow{{}'\mathrm{d}^{j-2}} & {}'A^{j-1} & \xrightarrow{{}'\mathrm{d}^{j-1}} & {}'A^j & \xrightarrow{{}'\mathrm{d}^j} \cdots \\
& \| & & \| & & \| & \\
& A^{j-2} & & A^{j-1} \bigoplus \ker \mathrm{d}^j & \xrightarrow{\tilde{\mathrm{d}}} & A^j & \\
& & \searrow^{\mathrm{d}^{j-2}} & \nearrow_{\iota} & & & \\
& & A^{j-1}, & & & &
\end{array} \tag{3.30}
$$

where we have put $\tilde{\mathrm{d}} := \mathrm{d}^{j-1} \oplus \iota$. Then ${}'A^\bullet$ is a subobject of $A^\bullet$. From (3.30) we have ${}'\mathrm{d}^{j-2} = \iota \circ \mathrm{d}^{j-2}$ and ${}'\mathrm{d}^{j-1} = \mathrm{d}^{j-1} \oplus \iota$, and the other ${}'\mathrm{d}^\bullet$ are the same as $\mathrm{d}^\bullet$. Since $\ker {}'\mathrm{d}^j = \ker \mathrm{d}^j$ and $\ker \mathrm{d}^j \subset \mathrm{im}\, {}'\mathrm{d}^{j-1} = \mathrm{im}(\mathrm{d}^j \oplus \iota)$ we get

$\mathrm{H}^j({}'A^\bullet) = \ker {}'\mathrm{d}^j \big/ \operatorname{im} {}'\mathrm{d}^{j-1} = 0$. Therefore,

$$\mathrm{H}^j(A^\bullet) = \mathrm{R}^j \mathrm{H}^0_{A^\bullet} A^\bullet = \mathrm{R}^j \ker(A^0 \xrightarrow{\mathrm{d}^0} A^1).$$

We will interpret the spectral sequence associated with a double complex as the spectral sequence induced by the composite functors as shown below.

$$\begin{array}{ccc} \mathsf{Co}^+(\mathsf{Co}^+(\mathscr{A})) & \overset{\mathrm{H}^0_\uparrow}{\rightsquigarrow} & \mathsf{Co}^+(\mathscr{A}) \\ & \searrow_{\mathrm{H}^0_\rightarrow \circ \mathrm{H}^0_\uparrow} & \downarrow \mathrm{H}^0_\rightarrow \\ & & \mathscr{A} \end{array} \tag{3.31}$$

Namely, $\mathrm{H}^0_\uparrow$ is the cohomology in the q-axis direction of a double complex as a complex of a complex in $\mathsf{Co}^+(\mathsf{Co}^+(\mathscr{A}))$,

$$C^{\bullet,\bullet} : \cdots \longrightarrow C^{\bullet,0} \longrightarrow C^{\bullet,1} \longrightarrow C^{\bullet,2} \longrightarrow \cdots \longrightarrow C^{\bullet,q} \longrightarrow \cdots .$$

Then for the left exact functor $\mathrm{H}^0_\uparrow$ and for the injective object $I^{\bullet,\bullet}$ of the category $\mathsf{Co}^+(\mathsf{Co}^+(\mathscr{A}))$,

$$\mathrm{H}^0_\uparrow(I^{\bullet,\bullet}) = \ker(I^{\bullet,0} \to I^{\bullet,1})$$

is an injective object of $\mathsf{Co}^+(\mathscr{A})$. We get the derived functors of $\mathrm{H}^0_\rightarrow$,

$$\mathrm{R}^j \mathrm{H}^0_\rightarrow(\mathrm{H}^0_\uparrow(I^{\bullet,\bullet})) = 0 \quad \text{for} \quad j \geq 1.$$

Therefore we can apply the spectral sequence of a composite functor to the diagram (3.31) obtaining

$${}'E_2^{p,q} = \mathrm{H}^p_\rightarrow(\mathrm{H}^q_\uparrow(C^{\bullet,\bullet})) = \mathrm{R}^p \mathrm{H}^0_\rightarrow(\mathrm{R}^q \mathrm{H}^0_\uparrow(C^{\bullet,\bullet})). \tag{3.32}$$

Next we pay attention to the abutment of this spectral sequence. From the definitions in (3.2) and (3.3) we have

$$\begin{aligned} C^0 &= C^{0,0} \\ C^1 &= C^{1,0} \oplus C^{0,1}, \end{aligned}$$

where $\mathrm{d}^0 : C^0 \to C^1$ is given by $\mathrm{d}^0 := \mathrm{d}^{0,0}_{(1,0)} \oplus \mathrm{d}^{0,0}_{(0,1)}$. That is,

$$(\mathrm{H}^0_\rightarrow \circ \mathrm{H}^0_\uparrow)(C^{\bullet,\bullet}) = \ker \mathrm{d}^0 = \mathrm{H}^0(C^\bullet),$$

where $C^\bullet$ is the complex defined by $C^n := \bigoplus_{p+q=n} C^{p,q}$ as in (3.2). Therefore, the spectral sequence (3.32) abuts upon $\mathrm{R}^n(\mathrm{H}^0_\rightarrow \circ \mathrm{H}^0_\uparrow)(C^{\bullet,\bullet}) = \mathrm{R}^n \mathrm{H}^0(C^\bullet)$, i.e., the total cohomology $\mathrm{H}^n(C^\bullet)$.

3.3.6 Hyperderived Functors

Let F be a left exact covariant additive functor from an abelian category $\mathscr{A}$ to an abelian category $\mathscr{B}$. Then for an object $A^\bullet$ of $\mathsf{Co}^+(\mathscr{A})$, $FA^\bullet$ is an object

$$\cdots \longrightarrow 0 \longrightarrow FA^0 \xrightarrow{F\mathrm{d}^0} FA^1 \xrightarrow{F\mathrm{d}^1} \cdots$$

of $\mathsf{Co}^+(\mathscr{B})$. This assignment induced by F is a functor Co^+F from $\mathsf{Co}^+(\mathscr{A})$ to $\mathsf{Co}^+(\mathscr{B})$. Then we get the following commutative diagram of categories and functors:

$$\begin{array}{ccc} \mathsf{Co}^+(\mathscr{A}) & \overset{\mathsf{Co}^+F}{\rightsquigarrow} & \mathsf{Co}^+(\mathscr{B}) \\ \big\downarrow \mathrm{H}^0 & \searrow^{\bar{F}} & \big\downarrow \mathrm{H}^0 \\ \mathscr{A} & \overset{F}{\rightsquigarrow} & \mathscr{B} \end{array} \tag{3.33}$$

where $(F \circ \mathrm{H}^0)A^\bullet = F(\ker \mathrm{d}^0)$ for an object $A^\bullet$ of $\mathsf{Co}^+(\mathscr{A})$. Since F is a left exact functor, we have

$$(F \circ \mathrm{H}^0)A^\bullet = F(\ker \mathrm{d}^0) = \ker(FA^0 \xrightarrow{F\mathrm{d}^0} FA^1). \tag{3.34}$$

Namely, $\mathrm{H}^0(\mathsf{Co}^+FA^\bullet) = F(\mathrm{H}^0A^\bullet)$ holds. We write the composite functor as $\bar{F}$. Since F and H^0 are left exact, $\bar{F}$ is a left exact functor from $\mathsf{Co}^+(\mathscr{A})$ to $\mathscr{B}$. We apply (3.26) to the spectral sequence associated with those composite functors in (3.34) obtaining:

$$\begin{aligned} E_2^{p,q} &= \mathrm{R}^p\mathrm{H}^0(\mathrm{R}^q\mathsf{Co}^+F)(A^\bullet) = \mathrm{H}^p((\mathrm{R}^q\mathsf{Co}^+F)(A^\bullet)) \\ {}'E_2^{p,q} &= \mathrm{R}^pF((\mathrm{R}^q\mathrm{H}^0)(A^\bullet)) = \mathrm{R}^pF(\mathrm{H}^q(A^\bullet)) \end{aligned} \tag{3.35}$$

abutting to $\mathrm{R}^n\bar{F}A^\bullet, n = p+q$. For those spectral sequences in (3.35) to exist we need to confirm the following. For an injective object $I^\bullet$ of $\mathsf{Co}^+(\mathscr{A})$, the higher derived functors of H^0 and F evaluated at $FI^\bullet$ in $\mathsf{Co}^+(\mathscr{B})$ and $\mathrm{H}^0(I^\bullet)$ in $\mathscr{A}$, respectively, must vanish. We will prove the corresponding, the clockwise and counter clockwise statements of the diagram (3.33). That is, we will confirm

$$\begin{cases} \mathrm{R}^p\mathrm{H}^0(FI^\bullet) = \mathrm{H}^p(FI^\bullet) = 0, p \geq 1 \\ \mathrm{R}^pF(\mathrm{H}^0I^\bullet) = \mathrm{R}^pF(\ker \mathrm{d}^0) = 0, p \geq 1. \end{cases} \tag{3.36}$$

The first assertion of (3.36) means that the complex $FI^\bullet$ is exact for $p \geq 1$. By the definition, $\ker(I^0 \xrightarrow{\mathrm{d}^0} I^1)$ is an injective object of $\mathscr{A}$. Then we get two injective resolutions of $\ker \mathrm{d}^0 = \ker(I^0 \xrightarrow{\mathrm{d}^0} I^1)$ in the sense of Section 2.6 in the previous Chapter. Namely, we have

$$\begin{array}{ccccccccc} & & I^0 & \xrightarrow{\mathrm{d}^0} & I^1 & \xrightarrow{\mathrm{d}^1} & I^2 & \xrightarrow{\mathrm{d}^2} & \cdots \\ & \overset{\epsilon}{\nearrow} & & & & & & & \\ \ker \mathrm{d}^0 & & & & & & & & \\ & \underset{\epsilon'}{\searrow} & & & & & & & \\ & & \ker \mathrm{d}^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{3.37}$$

From those injective resolutions of $\ker \mathrm{d}^0$, we get two complexes

$$\begin{array}{ccccccc} FI^0 & \xrightarrow{F\mathrm{d}^0} & FI^1 & \xrightarrow{F\mathrm{d}^1} & FI^2 & \xrightarrow{F\mathrm{d}^2} & \cdots \\ F\ker \mathrm{d}^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \end{array} \tag{3.38}$$

From the second sequence, $\mathrm{H}^p(F\ker \mathrm{d}^0 \to 0 \to 0 \to \cdots) = 0$ for $p \geq 1$. Namely,

$$\mathrm{H}^p(F\ker \mathrm{d}^0 \to 0 \to 0 \to \cdots) = \mathrm{R}^p F(\ker \mathrm{d}^0) = 0, \qquad \text{for } p \geq 1,$$

proving the second assertion. Since the derived functor is independent of the choice of injective resolution, the derived functor $\mathrm{R}^p F(\ker \mathrm{d}^0)$ can be computed via the first complex of (3.38). That is, $\mathrm{R}^p F(\ker \mathrm{d}^0) = \mathrm{H}^p(FI^\bullet) = 0$, for $p \geq 1$, proving the first assertion of (3.36).

Consequently, the abutment $\mathrm{R}^n \bar{F} A^\bullet$, where $\bar{F} = \mathrm{H}^0 \circ \mathsf{Co}^+ F = F \circ \mathrm{H}^0$, of the spectral sequences (3.35) is said to be the n-th *hyperderived functor of F evaluated at $A^\bullet$*. We often write $\mathrm{R}^n \bar{F} A^\bullet$ simply as $\mathrm{R}^n F A^\bullet$.

The derived functor $\mathrm{R}^q \mathsf{Co}^+ F A^\bullet$ of $\mathsf{Co}^+ F : \mathsf{Co}^+(\mathscr{A}) \rightsquigarrow \mathsf{Co}^+(\mathscr{B})$ is the complex

$$\mathrm{R}^q F A^0 \xrightarrow{\mathrm{R}^q F\mathrm{d}^0} \mathrm{R}^q F A^1 \xrightarrow{\mathrm{R}^q F\mathrm{d}^1} \mathrm{R}^q F A^2 \xrightarrow{\mathrm{R}^q F\mathrm{d}^2} \cdots$$

in $\mathsf{Co}^+(\mathscr{B})$. Namely, $(\mathrm{R}^q \mathsf{Co}^+ F)_{q \geq 0}$ satisfies (D.F.0) through (D.F.3) in Chapter II. Therefore, we can begin the spectral sequence $E_2^{p,q}$ in (3.35) from $E_1^{p,q}$:

$$\begin{array}{ccccccccc} E_1^{0,q} & \xrightarrow{\mathrm{d}_1^{0,q}} & E_1^{1,q} & \xrightarrow{\mathrm{d}_1^{1,q}} & \cdots & \xrightarrow{\mathrm{d}_1^{p-1,q}} & E_1^{p,q} & \xrightarrow{\mathrm{d}_1^{p,q}} & \cdots \\ \| & & \| & & & & \| & & \\ \mathrm{R}^q F A^0 & \xrightarrow{\mathrm{R}^q F\mathrm{d}^0} & \mathrm{R}^q F A^1 & \xrightarrow{\mathrm{R}^q F\mathrm{d}^1} & \cdots & \xrightarrow{\mathrm{R}^q F\mathrm{d}^{p-1}} & \mathrm{R}^q F A^p & \longrightarrow & \cdots \end{array} \tag{3.39}$$

i.e., $E_2^{p,q} = \mathrm{H}^p(E_1^{\bullet,q})$. That is, from the commutative diagram (3.33), we get the useful spectral sequences

$$\begin{cases} E_1^{p,q} = \mathrm{R}^q F A^p \\ {}'E_2^{p,q} = \mathrm{R}^p F(\mathrm{H}^q(A^\bullet)). \end{cases} \tag{3.40}$$

3.3.7 Hyperderived to Composite Functor

Let $\mathscr{A}$, $\mathscr{B}$, and $\mathscr{C}$ be abelian categories such that $\mathscr{A}$ and $\mathscr{B}$ have enough injectives. For left exact functors $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ and $G : \mathscr{B} \rightsquigarrow \mathscr{C}$, the diagram

$$\begin{array}{ccc} \mathscr{A} & \overset{F}{\rightsquigarrow} & \mathscr{B} \\ & \underset{G\circ F}{\searrow} & \downarrow{\scriptstyle G} \\ & & \mathscr{C} \end{array} \tag{3.41}$$

induces the commutative diagram

$$\begin{array}{ccc} \mathsf{Co}^+(\mathscr{B}) & \overset{\mathsf{Co}^+G}{\rightsquigarrow} & \mathsf{Co}^+(\mathscr{C}) \\ {\scriptstyle \mathrm{H}^0}\downarrow & \overset{\bar{G}}{\searrow} & \downarrow{\scriptstyle \mathrm{H}^0} \\ \mathscr{B} & \underset{G}{\rightsquigarrow} & \mathscr{C} \end{array} \tag{3.42}$$

Then for an object $A^\bullet = FI^\bullet$, where $I^\bullet$ is an injective resolution of A in $\mathscr{A}$, the spectral sequences in (3.40) become

$$\begin{cases} E_1^{p,q} = \mathrm{R}^q G A^p \\ {}'E_2^{p,q} = \mathrm{R}^p G(\mathrm{H}^p(A^\bullet)) \end{cases} \tag{3.43}$$

with the abutment $\mathrm{R}^n\bar{G}A^\bullet$, where $n = p + q$. The assumption (3.13), i.e., $\mathrm{R}^j G(FI) = 0$, $j > 0$, for an injective object I of $\mathscr{A}$ implies

$$E_1^{p,q} = \mathrm{R}^q G A^p = \mathrm{R}^q G(FI^p) = 0, \qquad \text{for } q > 1$$

in (3.43). Therefore, $E_2^{p,0}$ can be computed by taking the cohomology of

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & E_1^{p-1,0} & \longrightarrow & E_1^{p,0} & \longrightarrow & E_1^{p+1,0} & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & (G\circ F)I^{p-1} & \longrightarrow & (G\circ F)I^p & \longrightarrow & (G\circ F)I^{p+1} & \longrightarrow & \cdots \end{array} \tag{3.44}$$

Namely, $E_2^{p,0} = \mathrm{H}^p(E_1^{\bullet,0}) = \mathrm{H}^p(\mathrm{R}^0(G\circ F)I^\bullet) \approx \mathrm{H}^p((G\circ F)I^\bullet)$. By the definition (7.1) of the derived functor, we get $E_2^{p,0} = \mathrm{R}^p(G\circ F)A$. Consequently, we obtain $E_2^{p,0} = \mathrm{R}^p(G\circ F)A \approx E_\infty^{p,0} \approx E^p = \mathrm{R}^p\bar{G}A^\bullet$, the abutment. On the other hand, ${}'E_2^{p,q}$ in (3.43) can be computed as follows.

$${}'E_2^{p,q} = \mathrm{R}^p G(\mathrm{H}^q(FI^\bullet)) = \mathrm{R}^p G(\mathrm{R}^q F A)$$

abutting upon $E^n = \mathrm{R}^n\bar{G}A^\bullet \approx E_2^{n,0} = \mathrm{R}^n(G \circ F)A$. Namely, we get the spectral sequence of a composite functor from the hyperderived functor spectral sequences.

3.4 Cohomology of Sheaves over Topological Space

As in Section 1.10 in Chapter I, let T be a topological space and let $\mathscr{T}$ be the category of open sets of T. Let $\tilde{\mathscr{T}}$ be the full subcategory consisting of sheaves, as defined in Definition 5, of the category $\hat{\mathscr{T}}$ of presheaves over $\mathscr{T}$ to an abelian category $\mathscr{A}$. In Section 1.10 the stalk of $F \in \mathrm{Ob}(\tilde{\mathscr{T}}) \subset \mathrm{Ob}(\hat{\mathscr{T}})$ at $x \in T$ is defined as the direct limit $F_x = \varinjlim FU$, where the limit is taken over all open sets U such that $x \in U$. For a short exact sequence

$$0 \longrightarrow F' \xrightarrow{\phi} F \xrightarrow{\psi} F'' \longrightarrow 0 \tag{4.1}$$

of sheaves over $\mathscr{T}$, we have the induced sequence

$$F'U \xrightarrow{\phi_U} FU \xrightarrow{\psi_U} F''U \tag{4.2}$$

in the abelian category $\mathscr{A}$. We will prove that by taking the direct limit $\varinjlim$ (over U with $x \in U$) of the sequence (4.2) we obtain the short exact sequence (in $\mathscr{A}$)

$$0 \longrightarrow F'_x \xrightarrow{\phi_x} F_x \xrightarrow{\psi_x} F''_x \longrightarrow 0 \tag{4.3}$$

of stalks at x. Namely, we will show that

$$\varinjlim : \tilde{\mathscr{T}} \rightsquigarrow \mathscr{A}$$

is an exact functor. By the definition in Definition 5 in Chapter I, the induced morphism

$$\phi_x : F'_x \to F_x$$

is a monomorphism. Then we canonically get

$$0 \longrightarrow F'_x \xrightarrow{\phi_x} F_x \xrightarrow{\pi_x} \operatorname{coker} \phi_x \longrightarrow 0\,.$$

From the following four short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F'_x & \xrightarrow{\phi_x} & F_x & \xrightarrow{\pi_x} & \operatorname{coker}\phi_x & \longrightarrow & 0 \\
 & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & \varinjlim FU & \xrightarrow{\varinjlim \phi_U} & \varinjlim FU & \xrightarrow{\varinjlim \pi_U} & (\operatorname{coker}\phi)_x & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & F'_x & \xrightarrow{\phi_x} & F_x & \xrightarrow{\pi_x} & (\mathsf{sh}\operatorname{coker}\phi)_x & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \approx} & & \\
0 & \longrightarrow & F'_x & \xrightarrow{\phi_x} & F_x & \xrightarrow{\psi_x} & F''_x & \longrightarrow & 0
\end{array} \tag{4.4}$$

we get the exactness in (4.3). Note that $\operatorname{coker}\phi$ is the sheaf associated with $(\operatorname{coker}\phi)U = \operatorname{coker}\phi_U$ as in (14.9) in Chapter I, and $\mathsf{sh}\operatorname{coker}\phi$ is the sheaf associated with the presheaf $\operatorname{coker}\phi$ as in (15.1) also in Chapter I.

3.4.1 Left Exactness of Global Section Functor

For a sheaf F and an open set U of T we assign an object FU in $\mathscr{A}$. Namely, we have the functor

$$\tilde{\mathscr{T}} \times \mathscr{T} \rightsquigarrow \mathscr{A} \tag{4.5}$$

defined by

$$(F, U) \longmapsto FU \ .$$

This functor is covariant in $\tilde{\mathscr{T}}$ and contravariant in $\mathscr{T}$. For an open set U of T, the covariant functor induced by (4.5), $\cdot\, U : \tilde{\mathscr{T}} \rightsquigarrow \mathscr{A}$ is denoted by $\Gamma(U, \cdot)$. That is, $\Gamma(U, F) = FU$. Then for the exact sequence (4.1), we have the exact sequence

$$\begin{array}{ccccccc}
0 & \longrightarrow & \Gamma(U, F') & \longrightarrow & \Gamma(U, F) & \longrightarrow & \Gamma(U, F'') \\
 & & \| & & \| & & \| \\
0 & \longrightarrow & FU' & \longrightarrow & FU & \longrightarrow & F''U.
\end{array} \tag{4.6}$$

Namely, $\Gamma(U, \cdot) : \tilde{\mathscr{T}} \rightsquigarrow \mathscr{A}$ is a left exact functor, and is said to be a *global section functor*. One can prove the left exactness of $\Gamma(U, \cdot)$ by the exactness of (4.3) and the sheaf axiom (Sheaf) in Definition 5 in Chapter I. On the other hand, decompose the functor $\Gamma(U, \cdot)$ as

$$\begin{array}{ccc}
\tilde{\mathscr{T}} & \overset{\iota}{\rightsquigarrow} & \hat{\mathscr{T}} \\
 & {\scriptstyle \Gamma(U,\cdot)} \searrow \quad \swarrow {\scriptstyle (\cdot,U)} & \\
 & \mathscr{A}, &
\end{array} \tag{4.7}$$

where ι is the inclusion functor as in Section 1.15 and $(\cdot, U)$ is defined by:

$$\text{For} \quad F \in \text{Ob}(\hat{\mathscr{T}}), \qquad (\cdot, U)F = (F, U) := FU.$$

We have shown that in Section 1.14 the presheaf $(\ker \psi)U = \ker \psi_U$ of the sheaf morphism $F \xrightarrow{\psi} F''$ is a sheaf. Namely, $\Gamma(U, \ker \psi)$ is the kernel of $\psi_U : FU \to F''U$, i.e., $\ker(\iota F \to \iota F'')$ in $\hat{\mathscr{T}}$. Therefore, the composition of this left exact functor ι with the exact functor $(\cdot, U)$ is the left exact functor $\Gamma(U, \cdot)$. That is, for the exact sequence (4.1) of sheaves, the epimorphism ψ_x at each point does not guarantee the epimorphism of $\phi_U : FU \to F''U$.

3.4.2 Derived Functors of Global Section Functor

We need to show that the category $\tilde{\mathscr{T}}$ of sheaves over $\mathscr{T}$ to $\mathscr{A}$ is an abelian category. That is, we must verify (A.1) through (A.6) in Section 1.6 for $\tilde{\mathscr{T}}$. We will give an explanation (not a proof) for this fact. Let $\phi : F \to G$ be a morphism of sheaves in $\tilde{\mathscr{T}}$. We have already proved that the presheaf $\ker \phi_U = \ker(FU \xrightarrow{\phi_U} GU)$ is a sheaf. (See Section 1.14). Let $\operatorname{coker} \phi$, $\operatorname{im} \phi$ and $\operatorname{coim} \phi$ be the associated sheaves to the presheaves as defined in Section 1.14. For example, $\operatorname{coker} \phi = \mathsf{sh}(\text{presheaf } U \mapsto \operatorname{coker} \phi_U)$ in Subsection 1.15.1. Then, in $\mathscr{A}$, we have $(\ker \phi)_x = \varinjlim(\ker \phi_U) = \ker(\varinjlim \phi) = \ker \phi_x$, $(\operatorname{im} \phi)_x = \operatorname{im} \phi_x$, $(\operatorname{coker} \phi)_x = \operatorname{coker} \phi_x$ and $(\operatorname{coim} \phi)_x = \operatorname{coim} \phi_x$. By using the fact that a sheaf morphism $\phi : F \to G$ is determined locally, i.e., an isomorphism at each stalk $F_x \approx G_x$ induces an isomorphism of sheaves $F \xrightarrow{\approx} G$ (namely, the converse: (4.3) implying (4.1)), we obtain an isomorphism $\operatorname{coim} \phi \approx \operatorname{im} \phi$.

Let I be a sheaf so that $\operatorname{Hom}_{\tilde{\mathscr{T}}}(\cdot, I)$ is an exact functor from $\tilde{\mathscr{T}}$ to the category Ab of abelian groups. That is, I is an injective object in $\tilde{\mathscr{T}}$ satisfying the universal mapping property in Section 2.5. Then I is said to be an *injective sheaf*. The j-th *derived functor* of the *left exact functor* $\Gamma(U, \cdot) : \tilde{\mathscr{T}} \rightsquigarrow \mathscr{A}$ at $F \in \text{Ob}(\tilde{\mathscr{T}})$ is defined by

$$\mathrm{R}^j\Gamma(U, \cdot)F = \mathrm{H}^j(\Gamma(U, I^\bullet)) \tag{4.8}$$

where $I^\bullet$ is an injective resolution of F as in Section 2.6. Then the j-th derived functor defined by (4.8) is written as $\mathrm{H}^j(U, F)$ which is called the *j-th cohomology object* over U (the j-th *cohomology group* if $\mathscr{A} = $ Ab) with coefficient in F.

Next we will introduce another kind of a sheaf which plays the same role as far as cohomologies are concerned. A sheaf $\mathscr{F} \in \text{Ob}(\tilde{\mathscr{T}})$ is said to be a *flabby sheaf* if for open sets U and V satisfying $U \subset V$, the restriction morphism

$$\rho_U^V : \Gamma(V, \mathscr{F}) \longrightarrow \Gamma(U, \mathscr{F}) \tag{4.9}$$

is an epimorphism. Very important examples of flabby sheaves are the sheaves $\mathscr{B}$ and $\mathscr{C}$ of hyperfunctions and microfunctions, respectively. We will come back to the cohomological aspects of those sheaves in Chapter V. We will prove that for two resolutions of a sheaf $F \in \mathrm{Ob}(\tilde{\mathscr{T}})$; one by injective sheaves and the other by flabby sheaves:

$$\begin{cases} F \overset{\epsilon}{\to} I^\bullet \\ F \overset{\epsilon'}{\longrightarrow} \mathscr{F}^\bullet, \end{cases} \tag{4.10}$$

the induced complexes $\Gamma(U, I^\bullet)$ and $\Gamma(U, \mathscr{F}^\bullet)$ have isomorphic cohomologies. Namely, $\mathrm{H}^j(\Gamma(U, I^\bullet))$ and $\mathrm{H}^j(\Gamma(U, \mathscr{F}^\bullet))$ are isomorphic objects of $\mathscr{A}$. Namely, complexes $\Gamma(U, I^\bullet)$ and $\Gamma(U, \mathscr{F}^\bullet)$ are *quasi-isomorphic*. Re-write (3.33) as

$$\begin{array}{ccc} \mathsf{Co}^+(\tilde{\mathscr{T}}) & \xrightarrow{\ \mathsf{Co}^+\Gamma(U,\cdot)\ } & \mathsf{Co}^+(\mathscr{A}) \\ \Big\downarrow{\scriptstyle \mathcal{H}^0} & & \Big\downarrow{\scriptstyle \mathrm{H}^0} \\ \tilde{\mathscr{T}} & \xrightarrow{\ \Gamma(U,\cdot)\ } & \mathscr{A} \end{array} \tag{4.11}$$

Then apply the spectral sequences (3.40) to the above commutative diagram (4.11) to get

$$\begin{cases} E_1^{p,q} = \mathrm{R}^q\Gamma(U,\cdot)(\mathscr{F}^p) = \mathrm{H}^q(U, \mathscr{F}^p) \\ {}'E_2^{p,q} = \mathrm{R}^p\Gamma(U,\cdot)(\mathcal{H}^q(\mathscr{F}^\bullet)) = \mathrm{H}^p(U, \mathcal{H}^q(\mathscr{F}^\bullet)). \end{cases} \tag{4.12}$$

Note that the functor $\mathcal{H}^0 : \mathsf{Co}^+(\tilde{\mathscr{T}}) \rightsquigarrow \tilde{\mathscr{T}}$ is associated with the presheaf $\mathrm{H}^0(\mathscr{F}^\bullet(U)) = \ker(\mathscr{F}^0(U) \overset{\mathrm{d}_U^0}{\longrightarrow} \mathscr{F}^1(U))$, which is a left exact functor from $\mathsf{Co}^+(\tilde{\mathscr{T}})$ to $\tilde{\mathscr{T}}$ and that $\mathcal{H}^q(\mathscr{F}^\bullet)$ is the q-th derived functor of $\mathcal{H}^0$. Note also that $\mathcal{H}^q(\mathscr{F}^\bullet)$ may be regarded as the associated sheaf to the presheaf $\mathrm{H}^q(\mathscr{F}^\bullet(U)) = \ker(\mathscr{F}^q(U) \to \mathscr{F}^{q+1}(U)) \big/ \mathrm{im}(\mathscr{F}^{q-1}(U) \to \mathscr{F}^q(U))$. Since $\mathscr{F}^\bullet$ is acyclic, i.e., $\mathcal{H}^q(\mathscr{F}^\bullet) = 0$ for $q \geq 1$, we have ${}'E_2^{p,q} = 0$ for $q \geq 1$ in (4.12), and ${}'E^{p,0} = \mathrm{H}^p(U, \mathcal{H}^0(\mathscr{F}^\bullet)) \approx \mathrm{H}^p(U, F)$. By the definition of $\mathrm{H}^p(U, F)$ in (4.8) we have

$$\mathrm{H}^p(U, F) := \mathrm{R}^p\Gamma(U,\cdot)F := \mathrm{H}^p(\Gamma(U, I^\bullet)). \tag{4.13}$$

On the other hand, for each flabby sheaf $\mathscr{F}^p$, we have that $\mathrm{H}^q(U, \mathscr{F}^q) = 0$, for $q \geq 1$. (Afterwards we will give a sketch of the proof.) Then $E_1^{p,q} =$

$\mathrm{H}^q(U, \mathscr{F}^p) = 0$ for $q \geq 1$ in (4.12). That is, we have

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \Gamma(U, \mathscr{F}^{p-1}) & \longrightarrow & \Gamma(U, \mathscr{F}^{p}) & \longrightarrow & \Gamma(U, \mathscr{F}^{p+1}) & \longrightarrow & \cdots \\ & & \| & & \| & & \| & & \\ \cdots & \longrightarrow & E_1^{p-1,0} & \xrightarrow{\mathrm{d}_1^{p-1,0}} & E_1^{p,0} & \xrightarrow{\mathrm{d}_1^{p,0}} & E_1^{p+1,0} & \longrightarrow & \cdots \end{array} \tag{4.14}$$

Therefore, $E_2^{p,0}$ is the cohomology $\mathrm{H}^p(E_1^{\bullet,0})$, i.e., $E_2^{p,0} = \mathrm{H}^p(\Gamma(U, \mathscr{F}^\bullet))$. Since we have $E_2^{p,0} \approx E_3^{p,0} \approx \cdots \approx E_\infty^{p,0}$ and $E^p = \bigoplus_{p'+0=p} E_\infty^{p',0} \approx E_\infty^{p,0}$, we get $E_2^{p,0} = \mathrm{H}^p(\Gamma(U, \mathscr{F}^\bullet)) = E^p$. Consequently,

$$'E_2^{p,0} \approx \mathrm{H}^p(U, F) = \mathrm{H}^p(\Gamma(U, I^\bullet)) \approx E^p = \mathrm{H}^p(\Gamma(U, \mathscr{F}^\bullet)).$$

Namely, the derived functors

$$\mathrm{H}^p(U, F) = \mathrm{R}^p\Gamma(U, \cdot)F$$

can also be defined in terms of a flabby resolution of F.

Notes 17. (1) As observed in the above proof, $E_1^{p,q} = \mathrm{H}^q(U, \mathscr{F}^p) = 0$ for $q \geq 1$ implies the isomorphism between the derived functor $\mathrm{H}^p(U, F)$ and the cohomology $\mathrm{H}^p(\Gamma(U, \mathscr{F}^\bullet))$ of the complex $\Gamma(U, \mathscr{F}^\bullet)$. That is, any resolution $'I^\bullet$ of F satisfying $\mathrm{H}^q(U, 'I^p) = 0$ for $q \geq 1$ provides an isomorphism between $\mathrm{H}^p(U, F) = \mathrm{H}^p(\Gamma(U, I^\bullet))$ and $\mathrm{H}^p(\Gamma(U, 'I^\bullet))$. Such an object as $'I^p$ is said to be an F-*acyclic* object.

(2) In general, complexes $G^\bullet$ and $'G^\bullet$ of an abelian category are said to be *quasi-isomorphic* when their cohomologies $\mathrm{H}^q(G^\bullet)$ and $\mathrm{H}^q('G^\bullet)$ are isomorphic. Therefore, for quasi-isomorphic complexes G and $'G^\bullet$, the spectral sequences of hyperderived functors of a left exact functor F give the isomorphism

$$'E_2^{p,q}(G^\bullet) = \mathrm{R}^pF(\mathrm{H}^qG^\bullet) \approx {'E_2^{p,q}}('G^\bullet) = \mathrm{R}^pF(\mathrm{H}^q('G^\bullet)). \tag{4.15}$$

Consequently, their abutments, their hyperderived functors, $\mathrm{R}^nFG^\bullet$ and $\mathrm{R}^nF'G^\bullet$ are isomorphic. In particular, for quasi-isomorphic complexes of sheaves $G^\bullet$ and $'G^\bullet$, their *hypercohomologies of sheaves* $\mathrm{H}^n(U, G^\bullet)$ and $\mathrm{H}^n(U, 'G^\bullet)$ are isomorphic. Notice that if $I^\bullet$ and $\mathscr{F}^\bullet$ are resolutions of a sheaf F by injective sheaves and flabby sheaves, respectively, their hypercohomologies $\mathrm{H}^n(U, I^\bullet)$ and $\mathrm{H}^n(U, \mathscr{F}^\bullet)$ are isomorphic. This is because $\mathcal{H}^q(I^\bullet) = 0$ and $\mathcal{H}^q(\mathscr{F}^\bullet) = 0$ for $q \geq 1$. On the other hand, either from (3.40) or from (4.12), $E_1^{p,q}(I^\bullet) = \mathrm{H}^q(U, I^p) = 0$ and $E_1^{p,q}(\mathscr{F}^\bullet) = \mathrm{H}^q(U, \mathscr{F}^p) = 0$ for $q \geq 1$. Then their isomorphic abutments, the hypercohomologies, $\mathrm{H}^n(U, I^\bullet)$ and $\mathrm{H}^n(U, \mathscr{F}^\bullet)$ become the isomorphic cohomologies of complexes: $\mathrm{H}^n(\Gamma(U, I^\bullet)) = \mathrm{H}^n(U, F)$ and $\mathrm{H}^n(\Gamma(U, \mathscr{F}^\bullet))$, as shown in (1).

(3) We will sketch a proof of $\mathrm{H}^q(U, \mathscr{F}) = 0$, $q \geq 1$, for a flabby sheaf $\mathscr{F}$. Embed $\mathscr{F}$ into an injective sheaf I. Then we have the exact sequence of sheaves

$$0 \longrightarrow \mathscr{F} \xrightarrow{\iota} I \xrightarrow{\pi} I/\mathscr{F} \longrightarrow 0\,.$$

Then, the exactness of (4.3) implies that there exists W to obtain the epimorphism $I(W) \xrightarrow{\pi_W} (I/\mathscr{F})(W) \to 0$. For $W \subset V$, the flabbyness of $\mathscr{F}$ and also I implies the following commutative diagram

$$\begin{array}{ccccccc} 0 & & 0 & & 0 & & \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathscr{F}(W) & \xrightarrow{\iota_W} & I(W) & \xrightarrow{\pi_W} & (I/\mathscr{F})(W) & \longrightarrow & 0 \\ \uparrow & & \uparrow & & \uparrow & & \\ \mathscr{F}(V) & \xrightarrow{\iota_V} & I(V) & \xrightarrow{\pi_V} & (I/\mathscr{F})(W) & \longrightarrow & 0 \end{array} \tag{4.16}$$

Namely, $I/\mathscr{F}$ is also a flabby sheaf. Then for the exact sequence

$$0 \to \mathscr{F} \to I \to I/\mathscr{F} \to 0$$

of sheaves, (D.F.1) in Section 2.8 becomes the long exact sequence

$$\begin{array}{l} 0 \longrightarrow \Gamma(U, \mathscr{F}) \xrightarrow{\iota_U} \Gamma(U, I) \xrightarrow{\pi_U} \Gamma(U, I/\mathscr{F}) \longrightarrow \\ \longrightarrow \mathrm{H}^1(U, \mathscr{F}) \longrightarrow \mathrm{H}^1(U, I) \longrightarrow \mathrm{H}^1(U, I/\mathscr{F}) \longrightarrow \\ \longrightarrow \mathrm{H}^2(U, \mathscr{F}) \longrightarrow \mathrm{H}^2(U, I) \longrightarrow \mathrm{H}^2(U, I/\mathscr{F}) \longrightarrow \cdots \end{array} \tag{4.17}$$

Then one can prove that $\pi_U : \Gamma(U, I) \to \Gamma(U, I/\mathscr{F})$ is an epimorphism. Therefore, $\mathrm{H}^1(U, \mathscr{F}) = 0$. Since I is an injective object of $\tilde{\mathscr{T}}$ we have $\mathrm{H}^j(U, I) = \mathrm{R}^j\Gamma(U, \cdot)I = 0$ for $j \geq 1$. Hence, in (4.17) we get

$$\mathrm{H}^{j+1}(U, \mathscr{F}) \overset{\approx}{\longleftarrow} \mathrm{H}^j(U, I/\mathscr{F})$$

for $j \geq 1$. Since $I/\mathscr{F}$ is also flabby, the induction implies $\mathrm{H}^j(U, \mathscr{F}) = 0$ for $j \geq 1$.

3.4.3 Čech Cohomology

Let F be a presheaf over a topological space T, i.e., $F \in \mathrm{Ob}(\hat{\mathscr{T}})$, where $\hat{\mathscr{T}} = \mathscr{A}^{\mathscr{T}^\circ}$ as in Section 3.4. Namely, F is simply a contravariant functor from the category $\mathscr{T}$ induced by the topological space T to the abelian category $\mathscr{A}$. Let I be an index set. For each $i \in I$, let U_i be an open set of T, i.e., $U_i \in \mathrm{Ob}(\mathscr{T})$. Then the inclusion $\iota : U_i \hookrightarrow T$ induces the morphism $F(T) \xrightarrow{\rho^T_{U_i}} F(U_i)$ in $\mathscr{A}$ which is said to be the restriction morphism. Assume that $(U_i, i \in I)$ is a covering of T, i.e., $T = \cup_{i\in I} U_i$. For $\iota^{ij}_i : U_i \cap U_j \hookrightarrow U_i$, let us write $\rho^i_{ij} : F(U_i) \to F(U_i \cap U_j)$. Similarly, e.g.,

$$\rho^{ij}_{ijk} : F(U_{ij}) \to F(U_{ijk})$$

where $U_{ij} = U_i \cap U_j$ and $U_{ijk} = U_i \cap U_j \cap U_k$. Then we have the following sequence of restriction morphisms:

$$\prod F(U_i) \underset{\rho^i_{ij}}{\overset{\rho^j_{ij}}{\rightrightarrows}} \prod F(U_{ij}) \begin{array}{c} \xrightarrow{\rho^{jk}_{ijk}} \\ \xrightarrow{\rho^{ik}_{ijk}} \\ \xrightarrow{\rho^{ij}_{ijk}} \end{array} \prod F(U_{ijk}) \begin{array}{c} \xrightarrow{\rho^{jkl}_{ijkl}} \\ \xrightarrow{\rho^{ikl}_{ijkl}} \\ \xrightarrow{\rho^{ijl}_{ijkl}} \\ \xrightarrow{\rho^{ijk}_{ijkl}} \end{array} \cdots \tag{4.18}$$

Let $\mathrm{d}^0 := \rho^j_{ij} - \rho^i_{ij}$, and $\mathrm{d}^1 := \rho^{jk}_{ijk} - \rho^{ik}_{ijk} + \rho^{ij}_{ijk}$, e t c. Then, e.g., for $(f_i) \in \prod_{i\in I} F(U_i)$, we have

$$\mathrm{d}^0((f_i)) = \rho^j_{ij}(f_j) - \rho^i_{ij}(f_i)$$

and for $(f_{ij}) \in \prod_{i,j\in I} F(U_{ij})$, we have

$$\mathrm{d}^1((f_{ij})) = \rho^{jk}_{ijk}(f_{jk}) - \rho^{ik}_{ijk}(f_{ik}) + \rho^{ij}_{ijk}(f_{ij}).$$

In general, define

$$\begin{aligned} \mathrm{d}^n =& \rho^{i_1 i_2 \cdots i_{n+1}}_{i_0 i_1 \cdots i_{n+1}} - \rho^{i_0 i_2 \cdots i_{n+1}}_{i_0 i_1 \cdots i_{n+1}} + \cdots + (-1)^j \rho^{i_0 \cdots i_{j-1} i_{j+1} \cdots i_{n+1}}_{i_0 i_1 \cdots i_{n+1}} + \\ &+ \cdots + (-1)^{n+1} \rho^{i_0 i_1 \cdots i_n}_{i_0 i_1 \cdots i_{n+1}}. \end{aligned} \tag{4.19}$$

Let

$$C^j(U_i, i \in I; F) = C^j(\mathscr{U}, F) = \prod_{i_0,\ldots,i_j \in I} F(U_{i_0 \cdots i_j}),$$

where we, for ease of notation, write $\mathscr{U} := (U_i, i \in I)$. Then (4.18) becomes

$$C^0(\mathscr{U}, F) \xrightarrow{\mathrm{d}^0} C^1(\mathscr{U}, F) \xrightarrow{\mathrm{d}^1} C^2(\mathscr{U}, F) \xrightarrow{\mathrm{d}^2} \cdots \tag{4.20}$$

Since $\mathrm{d}^{j+1} \circ \mathrm{d}^j = 0$ is satisfied in (4.20), $C^\bullet(\mathscr{U}, F)$ is a complex which is said to be a *Čech complex*. The cohomology

$$\mathrm{H}^j(C^\bullet(\mathscr{U}, F)) := \ker \mathrm{d}^j \big/ \operatorname{im} \mathrm{d}^{j-1} \tag{4.21}$$

of (4.20) is said to be the *j-th Čech cohomology object* (*Čech cohomology group* if $\mathscr{A}$ is the category Ab of abelian groups) of the covering $\mathscr{U} = (U_i, i \in I)$ of T which is written as $\mathrm{H}^j(\mathscr{U}, T, F)$ or $\mathrm{H}^j(\mathscr{U}, F)$.

As we noted in (3.27), the j-th cohomology (4.21) is the j-th derived functor of the 0-th cohomology of the complex $C^\bullet(\mathscr{U}, F) = C^\bullet(U_i, i \in I; F)$. Let us study $\mathrm{H}^0(C^\bullet(\mathscr{U}, F))$. Namely, we compute $\ker \mathrm{d}^0 = \mathrm{H}^0(C^\bullet(\mathscr{U}, F))$ of (4.20) as follows. Let $(f_i) \in C^0(\mathscr{U}, F) = \prod F(U_i)$ satisfying

$$\mathrm{d}^0((f_i)) = \rho^j_{ij}(f_j) - \rho^i_{ij}(f_i) = 0$$

for $i, j \in I$. Therefore, if this presheaf F is a sheaf, then there exists a unique $f \in F(T)$ satisfying $\rho^T_i(f) = f_i$ for all $i \in I$. (See Definition 5 in Chapter I.) That is, we have the following diagram

$$\begin{array}{ccc} \tilde{\mathscr{T}} & \overset{\iota}{\rightsquigarrow} & \hat{\mathscr{T}} \\ & \searrow^{\Gamma(T,\cdot)} & \Big\downarrow \mathrm{H}^0(\mathscr{U},\cdot)=\mathrm{H}^0(C^\bullet(\mathscr{U},\cdot))=\ker \mathrm{d}^0 \\ & & \mathscr{A} \end{array} , \tag{4.22}$$

i.e., for $F \in \mathrm{Ob}(\tilde{\mathscr{T}})$, $\mathrm{H}^0(\mathscr{U}, \iota F) = \Gamma(T, F) = \mathrm{R}^0\Gamma(T, \cdot)F$. Recall that for an exact sequence $0 \to F' \to F \to F'' \to 0$ in $\tilde{\mathscr{T}}$, we only have the exact sequence $0 \to \iota F' \to \iota F \to \iota F''$ in $\hat{\mathscr{T}}$. Then for an injective sheaf I (an injective object of $\tilde{\mathscr{T}}$), we have $\mathrm{R}^q\mathrm{H}^0(\mathscr{U}, \cdot)\iota I = \mathrm{H}^q(\mathscr{U}, \iota I) = 0$ for $q \geq 1$ (which however requires a proof). We get the following induced spectral sequence from (3.26):

$$E_2^{p,q} = \mathrm{H}^p(\mathscr{U}, \mathrm{R}^q \iota F) \tag{4.23}$$

abutting to $\mathrm{H}^n(T, F) = \mathrm{R}^n\Gamma(T, \cdot)F$.

Next, suppose that $0 \to F' \to F \to F'' \to 0$ is an exact sequence of presheaves. We have the exact sequence $0 \to F'(U) \to F(U) \to F''(U) \to 0$ for an arbitrary open set U. Then we also have the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^j(\mathscr{U}, F') & \longrightarrow & C^j(\mathscr{U}, F) & \longrightarrow & C^j(\mathscr{U}, F'') & \longrightarrow & 0 \\ & & \Big\| & & \Big\| & & \Big\| & & \\ 0 & \longrightarrow & \prod F'(U_{i_0\cdots i_j}) & \longrightarrow & \prod F(U_{i_0\cdots i_j}) & \longrightarrow & \prod F''(U_{i_0\cdots i_j}) & \longrightarrow & 0 \end{array} \tag{4.24}$$

for an open set $U_{i_0\cdots i_j}$. Namely, we get the exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathscr{U}, F') \longrightarrow C^\bullet(\mathscr{U}, F) \longrightarrow C^\bullet(\mathscr{U}, F'') \longrightarrow 0 . \tag{4.25}$$

Then for (D.F.1) in Section 2.8 we get the following long exact sequence

$$
\begin{aligned}
&0 \longrightarrow \mathrm{H}^0(\mathscr{U}, F') \longrightarrow \mathrm{H}^0(\mathscr{U}, F) \longrightarrow \mathrm{H}^0(\mathscr{U}, F'') \longrightarrow \cdots . \\
&\cdots \longrightarrow \mathrm{H}^j(\mathscr{U}, F') \longrightarrow \mathrm{H}^j(\mathscr{U}, F) \longrightarrow \mathrm{H}^j(\mathscr{U}, F'') \longrightarrow \cdots
\end{aligned}
\tag{4.26}
$$

For coverings $\mathscr{U} = (U_i, i \in I)$ and $\mathscr{U}' = (U'_{i'}, i' \in I')$ of T, i.e., $T = \cup_{i\in I} U_i = \cup_{i' \in I'} U'_{i'}$, if there is a mapping $\rho : I' \to I$ satisfying $U'_{i'} \subset U_{\rho(i')}$ for all $i' \in I'$, $\mathscr{U}'$ is said to be a *refinement* of $\mathscr{U}$. Then the inclusion

$$U_{i'_0 i'_1 \cdots i'_j} = U_{i'_0} \cap U_{i'_1} \cap \cdots \cap U_{i'_j} \hookrightarrow U_{\rho(i'_0)\cdots\rho(i'_j)} = U_{\rho(i'_0)} \cap \cdots \cap U_{\rho(i'_j)} \tag{4.27}$$

induces the restriction morphism

$$F(U_{\rho(i'_0)\rho(i'_1)\cdots\rho(i'_j)}) \longrightarrow F(U_{i'_0 i'_1 \cdots i'_j}). \tag{4.28}$$

For a sequence of refinements

$$\mathscr{U} \longleftarrow \mathscr{U}' \longleftarrow \mathscr{U}'' \longleftarrow \cdots \tag{4.29}$$

we get the induced sequence of complexes and their cohomologies of these complexes

$$C^\bullet(\mathscr{U}, F) \longrightarrow C^\bullet(\mathscr{U}', F) \longrightarrow C^\bullet(\mathscr{U}'', F) \longrightarrow \cdots \tag{4.30}$$

and

$$\mathrm{H}^j(\mathscr{U}, F) \longrightarrow \mathrm{H}^j(\mathscr{U}', F) \longrightarrow \mathrm{H}^j(\mathscr{U}'', F) \longrightarrow \cdots , \tag{4.31}$$

respectively. Then define

$$\check{\mathrm{H}}^j(T, F) := \varinjlim (\, \mathrm{H}^j(\mathscr{U}, F) \longrightarrow \mathrm{H}^j(\mathscr{U}', F) \longrightarrow \cdots), \tag{4.32}$$

which is said to be the *j-th Čech cohomology object* of T of the presheaf F. Since we have $\mathrm{H}^0(\mathscr{U}, \iota F) = \Gamma(T, F)$, i.e., (4.22), for $j = 0$ we get

$$\check{\mathrm{H}}^0(T, \iota F) = \Gamma(T, F) \tag{4.33}$$

in the diagram

$$
\begin{array}{ccc}
\tilde{\mathscr{T}} & \overset{\iota}{\rightsquigarrow} & \hat{\mathscr{T}} \\
& {\scriptstyle \Gamma(T,\cdot)} \searrow & \downarrow {\scriptstyle \check{\mathrm{H}}^0(T,\cdot)} \\
& & \mathscr{A} .
\end{array}
\tag{4.34}
$$

By applying the spectral sequence (3.26) associated with composite functor to the above diagram, we have

$$E_2^{p,q} = \mathrm{R}^p\check{\mathrm{H}}^0(T,\cdot)(\mathrm{R}^q\iota F) = \mathrm{R}^p\check{\mathrm{H}}^0(T,\mathrm{R}^q\iota F) \tag{4.35}$$

abutting upon $\mathrm{H}^n(T,F)$. Note that the definition of the Čech cohomology of T at F is given as a direct limit, i.e., (4.32). On the other hand, $\mathrm{H}^p(\mathscr{U},\cdot)$ is defined as the cohomology of the complex $C^\bullet(\mathscr{U},F)$. Then, from (3.27) we have that the p-th cohomology $\mathrm{H}^p(\mathscr{U},\cdot)$ of the complex $C^\bullet(\mathscr{U},\cdot)$ is the p-th derived functor $\mathrm{R}^p\mathrm{H}^0(\mathscr{U},\cdot)$ of the 0-th cohomology $\mathrm{H}^0(\mathscr{U},\cdot)$. However, as we mentioned in (17) (3), $\mathrm{H}^p(\mathscr{U},\mathscr{F}) = 0$, $p \geq 1$ for an injective $\mathscr{F}$ (for a flabby sheaf). Consequently, since the direct limit is exact, $\mathrm{R}^p\check{\mathrm{H}}^0(T,\cdot)$ in the spectral sequence (4.35) coincides with the Čech cohomology $\check{\mathrm{H}}^p(T,\cdot)$. Namely, $\check{\mathrm{H}}^p(T,\cdot)$ becomes the derived functor. Let us re-write (4.35) as

$$E_2^{p,q} = \check{\mathrm{H}}^p(T,\mathrm{R}^q\iota F) \tag{4.36}$$

abutting upon $\mathrm{H}^n(T,F)$, $n = p+q$. The coefficient sheaf $\mathrm{R}^q\iota F$ in (4.36) can be computed as follows. Since ι is left exact, we have $\mathrm{R}^0\iota F \approx \iota F$ by (D.F.0) in Section 2.8. Therefore, for an open set $U \in \mathrm{Ob}(\mathscr{T})$,

$$\mathrm{R}^0\iota F(U) = \iota F(U) \approx F(U) \approx \Gamma(U,F).$$

Since $\mathrm{H}^p(U,F) = \mathrm{R}^p\Gamma(U,\cdot)F$, we get $\mathrm{R}^p\iota F(U) \approx \mathrm{H}^p(U,F)$.

Let us study the spectral sequence (4.23) to understand the spectral sequence in (4.35). The $E_2^{p,q}$-term of (4.23) is, by definition the (4.21), given by

$$E_2^{p,q} = \mathrm{H}^p(C^\bullet(\mathscr{U},\mathrm{R}^q\iota F)), \tag{4.37}$$

where $C^\bullet(\mathscr{U},\mathrm{R}^q\iota F) = \prod \mathrm{R}^q\iota F(U_{i_0i_1\cdots i_p})$. In the above, we computed

$$\mathrm{R}^q\iota F(U_{i_0i_1\cdots i_p})$$

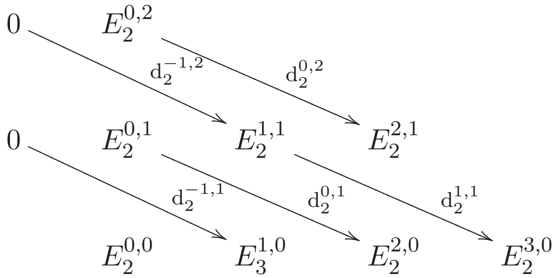
as $\mathrm{H}^q(U_{i_0i_1\cdots i_p},F)$. When $\mathrm{H}^q(U_{i_0i_1\cdots i_p},F) = 0$ for $q \geq 1$ we have $E_2^{p,q} = 0$ for $q \geq 1$ in (4.37). Then we get

$$0 = E_2^{p-2,1} \to E_2^{p,0} \to E_2^{p+2,-1} \to 0.$$

Consequently, we have $E_2^{p,0} \approx E_3^{p,0} \approx E_\infty^{p,0} \approx E^p \approx \mathrm{H}^p(T,F)$. That is, $E_2^{p,0} = \mathrm{H}^p(\mathscr{U},\mathrm{R}^0\iota F) \approx \mathrm{H}^p(\mathscr{U},\iota F) = \mathrm{H}^p(\mathscr{U},F) \approx \mathrm{H}^p(T,F)$. Summarizing: under the condition $\mathrm{H}^q(U_{i_0i_1\cdots i_p},F) = 0$ for $q \geq 1$, the Čech cohomology of the covering $\mathscr{U}$ coincides with the derived functor of the global section functor $\Gamma(T,\cdot)$, i.e., $\mathrm{H}^p(\mathscr{U},F) \approx \mathrm{H}^p(T,F)$. In particular, for $\mathrm{H}^q(U_{i_0i_1\cdots i_p},F) = 0$ for $q \geq 1$, the Čech cohomology in (4.32) of T is isomorphic to the derived functor, i.e., $\check{\mathrm{H}}^p(T,\mathrm{R}^0\iota F) \approx \check{\mathrm{H}}^p(T,F) \approx \mathrm{H}^p(T,F)$.

3.4.4 Edge Homomorphisms

As observed in the above, $E_2^{p,q}$-terms play a role in obtaining a morphism (or an isomorphism) between initial terms and abutments. Let us begin with $E_2^{p,q}$-terms of a first quadrant spectral sequence for $p = 0, 1, 2$ and $q = 0, 1, 2$ as follows:



(4.38)

Notice that the slope of $d_2^{p,q}$ is $-\frac{1}{2}$ as seen in Section 3.1. Since

$$0 \xrightarrow{d_2^{-2,1}} E_2^{0,0} \xrightarrow{d_2^{0,0}} 0 \qquad \text{and} \qquad 0 \xrightarrow{d_2^{-1,1}} E_2^{1,0} \xrightarrow{d_2^{1,0}} 0,$$

we have $E_2^{0,0} \approx E_\infty^{0,0} \approx E^0$ and $E_2^{1,0} \approx E_\infty^{1,0} \hookrightarrow E_\infty^{1,0} \oplus E_\infty^{0,1} = E^1$, respectively. Namely, we have the monomorphism $\iota_1 : E_2^{1,0} \hookrightarrow E^1$ given by $\iota_1(x_2^{1,0}) = (x_2^{1,0}, 0)$ for $x_2^{1,0} \in E_2^{1,0}$. Next, as for $E_2^{0,1}$ we have

$$0 \xrightarrow{d_2^{-2,2}} E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0}.$$

Hence, $E_3^{0,1} \approx \ker d_2^{0,1}$. Namely, we have $\ker d_2^{0,1} \approx E_3^{0,1} \hookrightarrow E_2^{0,1}$. Notice that we have $E_3^{0,1} \approx E_\infty^{0,1}$ since

$$0 \xrightarrow{d_3^{-3,3}} E_3^{0,1} \xrightarrow{d_3^{0,1}} 0.$$

Therefore we have $E^1 = E_\infty^{1,0} \oplus E_\infty^{0,1} \approx E_3^{1,0} \oplus E_3^{0,1} \xrightarrow{\pi_2} E_3^{0,1}$, where $\pi_2(x_3^{1,0}, x_3^{0,1}) = x_3^{0,1}$. Combining the above $\iota : E_3^{0,1} = \ker d_2^{0,1} \to E_2^{0,1}$ with $\pi_2 : E^1 \to E_3^{0,1}$, we get $E^1 \xrightarrow{\iota\circ\pi_2} E_3^{0,1}$. Next, for

$$E_2^{0,1} \xrightarrow{d_2^{0,1}} E_2^{2,0} \xrightarrow{d_2^{2,0}} 0$$

in (4.38), $E_3^{2,0}$ is the cohomology $E_2^{2,0}/\operatorname{im} d_2^{0,1}$. That is, we have the epimorphism $\pi : E_2^{2,0} \to E_3^{2,0}$. Then, as before, we have $E_3^{2,0} \approx E_\infty^{2,0}$. Since the abutment is $E^2 = E_\infty^{0,2} \oplus E_\infty^{1,1} \oplus E_\infty^{2,0}$, we get $E_3^{2,0} \xrightarrow{\iota_3} E^2$. The composition $\iota_3 \circ \pi$ is the morphism $E_2^{2,0} \xrightarrow{\iota_3\circ\pi} E^2$. Consequently, we obtain the following commutative diagram of initial terms and abutments:

$$
\begin{array}{ccccccc}
0 & & & & & & \\
\downarrow & & & & & & \\
E_2^{1,0} & & & & & & \\
\downarrow{\scriptstyle \iota_1} & & & & & & \\
E^1 & \xrightarrow{\iota\circ\pi_1} & E_2^{0,1} & \xrightarrow{\mathrm{d}_2^{0,1}} & E_2^{2,0} & \xrightarrow{\iota_3\circ\pi} & E^2 \\
\| & & \uparrow{\scriptstyle\iota} & & \downarrow{\scriptstyle\pi} & & \| \\
E_\infty^{0,1}\oplus E_\infty^{1,0} & & & & & & E_\infty^{0,2}\oplus E_\infty^{1,1}\oplus E_\infty^{2,0} \\
\approx\uparrow & & & & & \nearrow{\scriptstyle\iota_3} & \\
E_3^{0,1}\oplus E_3^{1,0} & \xrightarrow{\pi_1} & E_3^{0,1} & & E_\infty^{2,0} & &
\end{array}
\tag{4.39}
$$

For general p and q, we will study the edge terms $E_2^{p,0}$ and $E_2^{0,q}$ on the p-axis and q-axis, respectively. Let us begin with $E_2^{p,0}$ on the p-axis. Since we have

$$E_2^{p-2,1} \xrightarrow{\mathrm{d}_2^{p-2,1}} E_2^{p,0} \xrightarrow{\mathrm{d}_2^{p,0}} 0,$$

the cohomology at $E_2^{p,0}$ gives the natural epimorphisms

$$E_2^{p,0} \xrightarrow{\pi} E_3^{p,0} \xrightarrow{\pi} E_4^{p,0} \xrightarrow{\pi} \cdots E_p^{p,0} \xrightarrow{\pi} E_{p+1}^{p,0},$$

and beyond $E_{p+1}^{p,0}$ are the isomorphisms, i.e., $E_{p+1}^{p,0} \approx E_{p+2}^{p,0} \approx E_\infty^{p,0}$. By combining all those epimorphisms and isomorphisms with the monomorphism $\iota : E_\infty^{p,0} \hookrightarrow E^p$, we get the morphism from $E_2^{p,0}$ to the abutment:

$$E_2^{p,0} \xrightarrow{\iota\circ\pi^{p-1}} E^p \tag{4.40}$$

which is said to be the *edge morphism*. Next, as for $E_2^{0,q}$ on the q-axis, we have

$$0 \to E_2^{0,q} \xrightarrow{\mathrm{d}_2^{0,q}} E_2^{2,q-1}.$$

The cohomology $E_3^{0,q}$ at $E_2^{0,q}$ gives the monomorphism $\iota : E_3^{0,q} \hookrightarrow E_2^{0,q}$. By combining those induced monomorphisms with the isomorphisms

$$E_\infty^{0,q} \approx E_{q+3}^{0,q} \approx E_{q+2}^{0,q},$$

we have

$$E^q \xrightarrow{\iota^q\circ\pi} E_2^{0,q}, \tag{4.41}$$

where $\pi : E^q \to E_\infty^{0,q}$. The morphism (4.41) is the other edge morphism to $E_2^{0,q}$ on the q-axis from the abutment E^q.

The trivial cases of the edge morphisms applied to (4.36) are, e.g., at the edge (the origin) $E_2^{0,0}$, we have $0 \to E_2^{0,0} \to 0$. Since $E_2^{0,0} \approx E_\infty^{0,0} \approx E^0$ we get $E_2^{0,0} = \check{\mathrm{H}}^0(T, F) \approx E^0 = \mathrm{H}^0(T, F)$. Another example $0 \to E_2^{1,0} \xrightarrow{\iota_1} E^1$ becomes $0 \to \check{\mathrm{H}}^1(T, F) \to \mathrm{H}^1(T, F)$.

3.4.5 Relative Cohomology of Sheaves

Let U be an open set of a topological space T, i.e., $U \in \mathrm{Ob}(\mathscr{T})$ and let F be a sheaf over $\mathscr{T}$ to an abelian category $\mathscr{A}$ (or the category Ab of abelian groups), i.e., $F \in \mathrm{Ob}(\tilde{\mathscr{T}})$. Then for $V \hookleftarrow U$ we have the restriction morphism

$$\rho_U^V : F(V) = \Gamma(V, F) \to F(U) = \Gamma(U, F). \tag{4.42}$$

Define $\Gamma(V, U, F) := \ker \rho_U^V$. The following sequence

$$0 \longrightarrow \Gamma(V, U, F) \longrightarrow \Gamma(V, F) \xrightarrow{\rho_U^V} \Gamma(U, F) \tag{4.43}$$

is exact. When F is a flabby sheaf, ρ_U^V becomes epimorphic. (See (4.9).) Hence, F is flabby if and only if $\mathrm{H}^1(V, U, F) = 0$. This is because: for a flabby resolution of F, $F \to \mathscr{F}^\bullet$, the long exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(V, U, F) \longrightarrow \Gamma(V, F) \longrightarrow \Gamma(U, F) \longrightarrow \\ \longrightarrow \mathrm{H}^1(V, U, F) \longrightarrow \mathrm{H}^1(V, F) \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{H}^j(V, U, F) \longrightarrow \mathrm{H}^j(V, F) \longrightarrow \mathrm{H}^j(U, F) \longrightarrow \\ \longrightarrow \mathrm{H}^{j+1}(V, U, F) \longrightarrow \cdots \end{aligned} \tag{4.44}$$

is induced from the short exact sequence of complexes

$$0 \longrightarrow \Gamma(V, U, \mathscr{F}^\bullet) \longrightarrow \Gamma(V, \mathscr{F}^\bullet) \longrightarrow \Gamma(U, \mathscr{F}^\bullet) \longrightarrow 0\,. \tag{4.45}$$

Namely, $\mathrm{H}^j(V, U, F) = \mathrm{H}^j(\Gamma(V, U, \mathscr{F}^\bullet))$ and $\Gamma(V, U, \cdot)$ is a left exact functor from $\tilde{\mathscr{T}}$ to $\mathscr{A}$. That is, $\mathrm{H}^j(V, U, \cdot)$ is the derived functor of $\Gamma(V, U, \cdot)$ for $j \geq 0$. Then for an exact sequence of sheaves

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0 \tag{4.46}$$

we get the long exact sequence of relative cohomologies

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(V, U, F') \to \mathrm{H}^0(V, U, F) \to \mathrm{H}^0(V, U, F'') \longrightarrow \cdots \\ \cdots \longrightarrow \mathrm{H}^j(V, U, F') \to \mathrm{H}^j(V, U, F) \to \mathrm{H}^j(V, U, F'') \longrightarrow \cdots\,. \end{aligned} \tag{4.47}$$

Notice that for $W \supset V \supset U$ in T we have the exact sequence

$$\begin{aligned} 0 &\longrightarrow \mathrm{H}^0(W,V,F) \to \mathrm{H}^0(W,U,F) \to \mathrm{H}^0(V,U,F) \to \cdots \\ \cdots &\to \mathrm{H}^j(W,V,F) \to \mathrm{H}^j(W,U,F) \to \mathrm{H}^j(V,U,F) \to \cdots \end{aligned} \tag{4.48}$$

generalizing (4.44). Furthermore, for a closed set C in T satisfying $C \subset U$, the induced morphism from the restriction becomes the excision isomorphism

$$\mathrm{H}^j(T,U,F) \xrightarrow{\approx} \mathrm{H}^j(T-C,U-C,F). \tag{4.49}$$

For open sets U and U' we also have the Mayer–Vietoris sequence:

$$\begin{aligned} 0 &\to \mathrm{H}^0(T,U\cup U',F) \to \begin{matrix}\mathrm{H}^0(T,U,F)\\ \bigoplus \\ \mathrm{H}^0(T,U',F)\end{matrix} \to \mathrm{H}^0(T,U\cap U',F) \to \cdots \\ \cdots &\to \mathrm{H}^j(T,U\cup U',F) \to \begin{matrix}\mathrm{H}^j(T,U,F)\\ \bigoplus \\ \mathrm{H}^j(T,U',F)\end{matrix} \to \mathrm{H}^j(T,U\cap U',F) \to \cdots \end{aligned} \tag{4.50}$$

Even more generally, for $U \subset V$ and $U' \subset V'$, we have: for $j \geq 0$,

$$\cdots \to \mathrm{H}^j(V\cup V',U\cup U',F) \longrightarrow \begin{matrix}\mathrm{H}^j(V',U,F)\\ \bigoplus \\ \mathrm{H}^j(V',U',F)\end{matrix} \longrightarrow$$

$$\longrightarrow \mathrm{H}^j(V\cap V',U\cap U',F) \to \cdots .$$

3.4.6 Spectral Sequences and Relative Hypercohomologies

Let $F^\bullet$ be a complex of sheaves. Namely,

$$F^0 \xrightarrow{\mathrm{d}^0} F^1 \xrightarrow{\mathrm{d}^1} F^2 \xrightarrow{\mathrm{d}^2} \cdots$$

is a sequence of sheaves and morphisms of (pre-) sheaves satisfying

$$\mathrm{d}^j \circ \mathrm{d}^{j-1} = 0, \qquad \text{for } j \geq 1.$$

The sheaf version of the commutative diagram (3.33) becomes

$$\begin{array}{ccc} \mathsf{Co}^+(\tilde{\mathscr{T}}) & \overset{\mathcal{H}^0}{\rightsquigarrow} & \tilde{\mathscr{T}} \\ \Big\downarrow{\scriptstyle \mathrm{H}^0(T,U,\cdot)} & & \Big\downarrow{\scriptstyle \mathrm{H}^0(T,U,\cdot)} \\ \mathsf{Co}^+(\mathscr{A}) & \overset{\mathrm{H}^0}{\rightsquigarrow} & \mathscr{A} \end{array} \tag{4.51}$$

where for a complex $F^\bullet$ of sheaves in $\mathsf{Co}^+(\tilde{\mathscr{T}})$, $\mathcal{H}^0(F^\bullet)$ is the associated 0-th cohomology sheaf of $F^\bullet$. Then we have

$$\ker(\mathrm{H}^0(T,U,F^0) \to \mathrm{H}^0(T,U,F^1)) = \mathrm{H}^0(T,U,\mathcal{H}^0(F^\bullet)), \tag{4.52}$$

which is

$$\begin{aligned} &= \{f \in \Gamma(T,F^0) \mid \rho_U^T(f) = 0, \mathrm{d}^0(f) = 0 \text{ in } \Gamma(T,F^1)\} = \\ &= \mathrm{H}^0(T,U,\ker(F^0 \xrightarrow{\mathrm{d}^0} F^1)). \end{aligned} \tag{4.53}$$

The commutativity of (4.51) means the equality of (4.52). The spectral sequences in Subsection 3.3.6 corresponding to the diagram (4.51) become

$$\begin{aligned} E_2^{p,q} &= \mathrm{H}^p((\mathrm{H}^q(T,U,F^j))_{j\geq 0}), \text{ or} \\ E_1^{p,q} &= \mathrm{H}^q(T,U,F^p) \end{aligned} \tag{4.54a}$$

and

$$'E_2^{p,q} = \mathrm{H}^p(T,U,\mathcal{H}^q(F^\bullet)) \tag{4.54b}$$

abutting upon the *relative hypercohomology* $\mathrm{H}^n(T,U,F^\bullet)$ *with coefficient in the complex* $F^\bullet$ *of sheaves*. Notice also that

$$\begin{array}{ccccccccc} \cdots \longrightarrow & E_1^{p-1,q} & \xrightarrow{\mathrm{d}_1^{p-1,q}} & E_1^{p,q} & \xrightarrow{\mathrm{d}_1^{p,q}} & E_1^{p+1,q} & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ \cdots \longrightarrow & \mathrm{H}^q(T,U,F^{p-1}) & \longrightarrow & \mathrm{H}^q(T,U,F^p) & \longrightarrow & \mathrm{H}^q(T,U,F^{p+1}) & \longrightarrow \cdots . \end{array}$$

Remarks 2. (1) For a given $K \in \mathrm{Ob}(\tilde{\mathscr{T}})$, if a complex $F^\bullet$ is a cyclic resolution of K, i.e., $\mathcal{H}^j(F^\bullet) = 0$ for $j \geq 1$ and $K = \ker(F^0 \to F^1)$, from (4.54b) we have $'E_2^{p,q} = \mathrm{H}^p(T,U,\mathcal{H}^q(F^\bullet)) = 0$ for $q \geq 1$ and $'E_2^{p,0} = \mathrm{H}^p(T,U,K)$. Consequently we get $'E_2^{p,0} \approx {}'E_\infty^{p,0} \approx E^p$, i.e., $\mathrm{H}^p(T,U,K) \approx \mathrm{H}^p(T,U,F^\bullet)$.

(2) If $F^\bullet$ and $G^\bullet$ are quasi-isomorphic, the isomorphism $\mathcal{H}^q(F^\bullet) \approx \mathcal{H}(G^\bullet)$ induces the isomorphisms on the $'E_2^{p,q}$-terms and the abutments

$$\mathrm{H}^n(T,U,F^\bullet) \approx \mathrm{H}^n(T,U,G^\bullet).$$

(See Note 17 (2).)

(3) When $\mathrm{H}^q(T,U,F^p) = 0$ for $q \geq 1$, the spectral sequence in (4.54a) becomes $E_1^{p,q} = \mathrm{H}^q(T,U,F^p) = 0$, $q \geq 1$. Namely, we have

$$E_2^{p,0} = \mathrm{H}^p(E_1^{\bullet,0}) = \mathrm{H}^p(\mathrm{H}^0(T,U,F^\bullet)).$$

Therefore, the relative hypercohomology $\mathrm{H}^n(T, U, F^\bullet)$, the abutment, is isomorphic to

$$E_2^{n,0} = \mathrm{H}^n(\Gamma(T, U, F^\bullet)).$$

In particular, when $F^\bullet \in \mathrm{Ob}(\mathsf{Co}^+(\tilde{\mathscr{T}}))$ is a cyclic resolution of K satisfying $\mathrm{H}^q(T, U, F^p) = 0$ for $q \geq 1$, then the induced morphism from $E_2^{n,0}$ to the abutment is the natural isomorphism

$$\mathrm{H}^n(T, U, K) \approx \mathrm{H}^n(\mathrm{H}^0(T, U, F^\bullet)). \tag{4.55}$$

For example, let T be a differentiable manifold and let $\Omega_T^\bullet$ be the complex of the sheaves Ω_T^p of germs of p-forms on T. Then the De Rham-Theorem states that the cohomology of the complex $\Gamma(T, \Omega_T^\bullet)$ of global sections, the abutment, and the cohomology with coefficient in the constant sheaf $\mathcal{R} := \ker(\Omega_T^0 \xrightarrow{\mathrm{d}^0} \Omega_T^1)$ are isomorphic, i.e.,

$$\mathrm{H}^n(\Gamma(T, \Omega_T^\bullet)) \approx \mathrm{H}^n(T, \Omega_T^\bullet) \approx \mathrm{H}^n(T, \mathcal{R}). \tag{4.56}$$

3.4.7 Leray Spectral Sequence

Let $f : T \to S$ be a continuous map of topological spaces T and S and let $\tilde{\mathscr{T}}$ and $\tilde{\mathscr{S}}$ be the categories of sheaves over T and S, respectively. Then we will define a functor

$$f_* : \tilde{\mathscr{T}} \to \tilde{\mathscr{S}} \tag{4.57}$$

as follows. For a sheaf F over T, define

$$f_*F(V) := F(f^{-1}(V)) \tag{4.58}$$

where V is an open set of S. Since $f^{-1}(V \cap V') = f^{-1}(V) \cap f^{-1}(V')$, the presheaf $f_*F(V)$ defined by (4.58) becomes a sheaf, i.e., condition (Sheaf) in Definition 5 in Chapter I is satisfied. Note that $f_* : \tilde{\mathscr{T}} \rightsquigarrow \tilde{\mathscr{S}}$ is a left exact functor since $\Gamma(f^{-1}(V), \cdot)$ is left exact, i.e., (4.6). Therefore,

$$\mathrm{H}^0(f^{-1}(V), F) \approx \Gamma(f^{-1}(V), F).$$

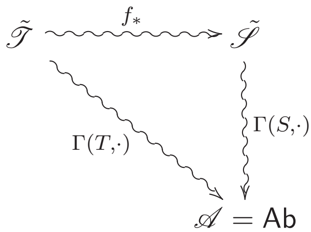
Then the derived functor $\mathrm{H}^j(f^{-1}(V), F)$ is a presheaf over S. Define $\mathrm{R}^j f_*F$ as the associated sheaf to this presheaf:

$$V \rightsquigarrow \mathrm{H}^j(f^{-1}(V), F). \tag{4.59}$$

The notation $\mathrm{R}^j f_*$, the derived functor of f_* is supported by the facts that $\mathrm{R}^0 f_* \approx f_*$ as functors and that for an injective sheaf $\mathcal{I}_S$ over S,

$$\mathrm{R}^j f_* \mathcal{I}_S = 0,$$

i.e., $(\mathrm{R}^j f_* \mathfrak{I}_S)_y = \varinjlim_{y \in V} \mathrm{H}^j(f^{-1}(V), \mathfrak{I}_S) = 0$. Consider the following commutative diagram:

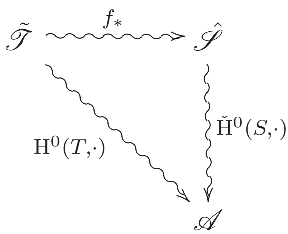


(4.60)

where $\Gamma(S, f_*F) = \Gamma(f^{-1}(S), F) = \Gamma(T, F)$. Then for an injective sheaf $\mathfrak{I}_T$ over T we have that $f_*\mathfrak{I}_T$ is a $\Gamma(S, \cdot)$-acyclic object of $\tilde{\mathscr{S}}$. (See Notes 17 (1).) Namely, $\mathrm{R}^q\Gamma(S, \cdot)(f_*\mathfrak{I}_T) = \mathrm{H}^q(f^{-1}(S), \mathfrak{I}_T) = \mathrm{H}^q(T, \mathfrak{I}_T) = 0$ for $q \geq 1$. Therefore, from (3.26) we have the following spectral sequence of a composite functor:

$$E_2^{p,q} = \mathrm{H}^p(S, \mathrm{R}^q f_* F) \tag{4.61}$$

abutting upon $E^n = \mathrm{H}^n(T, F)$, $n = p + q$. This spectral sequence is said to be the *Leray spectral sequence* induced by $f : T \to S$. The derived functor $\mathrm{R}^q f_* F$ is said to be the *higher direct image* of the *direct image* f_*F of F by f. Furthermore, by considering f_* as a functor from $\tilde{\mathscr{T}}$ to the category $\hat{\mathscr{S}}$ of presheaves over S, the diagram



(4.62)

implies the spectral sequence

$$E_2^{p,q} = \check{\mathrm{H}}^p(S, (\mathrm{R}^q f_*)_{\text{pre-sh}}(F)) \tag{4.63}$$

abutting upon $\mathrm{H}^n(T, F)$. Similarly, for $\mathrm{H}^0(\mathscr{U}, \cdot) : \hat{\mathscr{S}} \rightsquigarrow \mathscr{A}$ instead of $\check{\mathrm{H}}^0(S, \cdot)$ we get

$$E_2^{p,q} = \mathrm{H}^p(\mathscr{U}, (\mathrm{R}^q f_*)_{\text{pre-sh}}(F)) \tag{4.64}$$

with abutment $\mathrm{H}^n(T, F)$, where $\mathscr{U}$ is a covering of S.

We will generalize (4.61) to the relative hypercohomology case. So, let $f : T \to S$ be a continuous map of topological spaces and let U and V be open sets of T and S, respectively. For $F^\bullet \in \mathrm{Ob}(\mathsf{Co}^+(\tilde{\mathscr{T}}))$ let us consider the presheaf over S defined by

$$W \rightsquigarrow \mathrm{H}^q(f^{-1}(W), f^{-1}(W) \cap U, F^\bullet). \tag{4.65}$$

Let $\mathrm{R}^q f_{*,S,V}(T, U, F^\bullet)$ be the associated sheaf to the presheaf defined by (4.65). We have the following generalized Leray spectral sequence:

$$E_2^{p,q} = \mathrm{H}^p(S, V, \mathrm{R}^q f_{*,S,V}(T, U, F^\bullet)) \tag{4.66}$$

abutting upon $\mathrm{H}^n(T, f^{-1}(V) \cup U, F^\bullet)$. This spectral sequence is said to be the *second Leray spectral sequence of relative hypercohomology*. See the following diagram for (4.66):

$$\begin{array}{ccc} \mathsf{Co}^+(\tilde{\mathscr{T}}) & \xrightarrow{\mathrm{R}^0 f_{*,S,V}(T,U,\cdot)} & \mathsf{Co}^+(\tilde{\mathscr{S}}) \\ & \searrow^{\Gamma(T,f^{-1}(V)\cup U,\cdot)} & \downarrow{\scriptstyle \Gamma(S,V,\cdot)} \\ & & \mathscr{A}. \end{array} \tag{4.67}$$

See S. Lubkin and G. Kato, *Second Leray spectral sequence of relative hypercohomology*, Proc. Nat. Acad. Sci. U.S.A **75** (1978), no 10, 4666–4667.

3.5 Higher Derived Functors of $\varprojlim$

In Section 1.8 we defined an inverse limit of a covariant functor $F : \mathscr{C}' \rightsquigarrow \mathscr{C}$, i.e., $F \in \mathrm{Ob}(\mathscr{C}^{\mathscr{C}'})$. In this Section we consider the case $\mathscr{C}' = \mathbb{Z}$, where $i \xrightarrow{\phi} j$ for $i \geq j$ in $\mathbb{Z}$. Then in $\mathscr{C}$ for $F \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}})$, we have $Fi \to Fj$, which will be written as $F^i \xrightarrow{F\phi} F^j$ in this Section. First of all, let $\mathscr{C}$ be an abelian category. In what will follow we will define the derived functor of the inverse limit $\varprojlim$ which is a functor from $\mathscr{C}^{\mathbb{Z}}$ to $\mathscr{C}$.

Secondly, assume that the direct product $\prod_{i \in \mathbb{Z}} F^i$ exists in $\mathscr{C}$ where

$$\prod : \mathscr{C}^{\mathbb{Z}} \rightsquigarrow \mathscr{C}$$

is a functor. Using an exact embedding in Subsection 1.6.1; $(a_i) \in \prod_{i \in \mathbb{Z}} F^i$ belongs to $\varprojlim F^i$ if $a_j = F\phi(a_i)$ for all $i \xrightarrow{\phi} j$. (See Section 1.8).

Now we compute the derived functor $\mathrm{R}^j \varprojlim$ of

$$\varprojlim : \mathscr{C}^{\mathbb{Z}} \rightsquigarrow \mathscr{C} \tag{5.1}$$

by constructing a complex

$$C^\bullet : C^0 \xrightarrow{\mathrm{d}^0} C^1 \xrightarrow{\mathrm{d}^1} \cdots$$

so that $\mathrm{H}^0(C^\bullet) = \ker \mathrm{d}^0 = \varprojlim F^i$. Consequently we get

$$\mathrm{H}^j(C^\bullet) = \mathrm{R}^j\mathrm{H}^0(C^\bullet) = \mathrm{R}^j \varprojlim F^i. \tag{5.2}$$

Here is a construction of $C^\bullet$: for $F = (F^i) \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}})$ define

$$\begin{cases} C^0 = \prod_{i\in\mathbb{Z}} F^i \\ C^1 = \prod_{i\in\mathbb{Z}} F^i \\ C^j = 0, \quad \text{for } j = 2, 3, \ldots, \end{cases} \tag{5.3}$$

where $\mathrm{d}^0 : C^0 \to C^1$ is defined by

$$\pi^i \circ \mathrm{d}^0 = F\phi \circ \pi^{i+1} - \pi^i \tag{5.4}$$

in the diagram

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & C^0 & \xrightarrow{\mathrm{d}^0} & C^1 & \xrightarrow{\mathrm{d}^1} & 0 \longrightarrow \cdots \\ & & & & {\scriptstyle\pi^{i+1}}\downarrow & \searrow^{\pi} & \downarrow{\scriptstyle\pi_i} & & \\ \cdots & \longrightarrow & & & F^{i+1} & \longrightarrow & F^i & \longrightarrow & \cdots \end{array} \tag{5.5}$$

and π being the projection, i.e., $\pi^i((a_i)) = a_i$. Namely, for $(a_i) \in C^0$

$$\mathrm{d}^0((a_i)) = (\mathrm{d}^0_i(a_i)) = (F\phi(a_{i+1}) - a_i) \in C^1 = \prod_{i\in\mathbb{Z}} F^i.$$

Then we get

$$\begin{aligned} \mathrm{H}^0(C^\bullet) = \ker \mathrm{d}^0 = \{(a_i) \in C^0 \mid \mathrm{d}^0(a_i) = (0_i)\} = \\ = \{(a_i) \in C^0 \mid F\phi(a_{i+1}) = a_i\} = \\ = \varprojlim F^i \subset C^0 = \prod_{i\in\mathbb{Z}} F^i. \end{aligned} \tag{5.6}$$

Next,

$$\begin{array}{ccccc} \mathrm{H}^1(C^\bullet) & = & \mathrm{R}^1\mathrm{H}^0(C^\bullet) & = & \mathrm{R}^1 \varprojlim F^i \\ \| & & & & \\ \ker \mathrm{d}^1 / \operatorname{im} \mathrm{d}^0 & = & C^1 / \operatorname{im} \mathrm{d}^0 & = & \operatorname{coker} \mathrm{d}^0, \end{array} \tag{5.7}$$

and

$$\mathrm{H}^j(C^\bullet) = \mathrm{R}^j \varprojlim F^i = 0, \quad \text{for } j \geq 2. \tag{5.8}$$

We often write $\varprojlim^{(j)}$ or $\varprojlim^{j}$ for the higher derived functors $\mathrm{R}^j \varprojlim$ of $\varprojlim$ from $\mathscr{C}^{\mathbb{Z}}$ to $\mathscr{C}$.

Furthermore, let us assume that the functor

$$\prod : \mathscr{C}^{\mathbb{Z}} \rightsquigarrow \mathscr{C}$$

is exact. Then, for a short exact exact sequence in $\mathscr{C}^{\mathbb{Z}}$

$$0 \longrightarrow {}'F \longrightarrow F \longrightarrow {}''F \longrightarrow 0,$$

namely

$$0 \longrightarrow ({}'F^i) \longrightarrow (F^i) \longrightarrow ({}''F^i) \longrightarrow 0,$$

we get the exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \prod {}'F^i & \longrightarrow & \prod F^i & \longrightarrow & \prod {}''F^i & \longrightarrow & 0 \\
 & & \big\| \scriptstyle\mathrm{def} & & \big\| \scriptstyle\mathrm{def} & & \big\| \scriptstyle\mathrm{def} & & \\
0 & \longrightarrow & {}'C^\bullet & \longrightarrow & C^\bullet & \longrightarrow & {}''C^\bullet & \longrightarrow & 0.
\end{array} \tag{5.9}$$

in $\mathscr{C}$. Following (5.4) we have the induced diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & & \\
 & & {\scriptstyle {}'\mathrm{d}^1}\uparrow & & {\scriptstyle \mathrm{d}^1}\uparrow & & {\scriptstyle {}''\mathrm{d}^1}\uparrow & & \\
0 & \longrightarrow & {}'C^1 & \longrightarrow & C^1 & \longrightarrow & {}''C^1 & \longrightarrow & 0 \\
 & & {\scriptstyle {}'\mathrm{d}^0}\uparrow & & {\scriptstyle \mathrm{d}^0}\uparrow & & {\scriptstyle {}''\mathrm{d}^0}\uparrow & & \\
0 & \longrightarrow & {}'C^0 & \longrightarrow & C^0 & \longrightarrow & {}''C^0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{5.10}$$

Then from (5.10) we have the following exact sequence

$$\begin{array}{l}
0 \longrightarrow \ker {}'\mathrm{d}^0 \longrightarrow \ker \mathrm{d}^0 \longrightarrow \ker {}''\mathrm{d}^0 \longrightarrow \\
\longrightarrow \operatorname{coker} {}'\mathrm{d}^0 \longrightarrow \operatorname{coker} \mathrm{d}^0 \longrightarrow \operatorname{coker} {}''\mathrm{d}^0 \longrightarrow 0.
\end{array} \tag{5.11}$$

Note that the above assertion, i.e., (5.10) implies (5.11), is referred to as the *Snake Lemma*. That is, (5.11) becomes the exact sequence

$$\begin{aligned} 0 \longrightarrow \varprojlim {}'F^i \longrightarrow \varprojlim F^i \longrightarrow \varprojlim {}''F^i \longrightarrow \\ \longrightarrow \varprojlim^{(1)} {}'F^i \longrightarrow \varprojlim^{(1)} F^i \longrightarrow \varprojlim^{(1)} {}''F^i \longrightarrow 0 \end{aligned} \tag{5.12}$$

indicating the left exactness of $\varprojlim$ and the right exactness of $\varprojlim^{(1)}$.

3.5.1 Cohomology and Inverse Limit

An *inverse system* from $\mathbb{Z}$ to $\mathscr{C}$ is a covariant functor F from $\mathbb{Z}$ to an abelian category $\mathscr{C}$ as in Section 3.5, i.e., $F \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}})$. In this section we consider the case where $\mathscr{C}$ is replaced by the category $\mathsf{Co}^+(\mathscr{C})$ of complexes of $\mathscr{C}$. Namely, for each $i \in \mathbb{Z}$, $F_i^\bullet$ is a complex satisfying:

$$F_i^\bullet \xrightarrow{F^\bullet \phi_j^i} F_j^\bullet, \qquad \text{for all} \quad i \xrightarrow{\phi_j^i} j \text{ in } \mathbb{Z}, \tag{5.13}$$

and for each $i \in \mathbb{Z}$,

$$F_i^\bullet : \cdots \xrightarrow{\mathrm{d}_i^{p-1}} F_i^p \xrightarrow{\mathrm{d}_i^p} F_i^{p+1} \xrightarrow{\mathrm{d}_i^{p+1}} \cdots \tag{5.14}$$

satisfies $\mathrm{d}_i^{p+1} \circ \mathrm{d}_i^p = 0$ for all $p \geq 0$. That is, we are considering an object of the category $\mathsf{Co}^+(\mathscr{C})^{\mathbb{Z}}$.

Let us considering the following first quadrant double complex with only first and second non-zero rows:

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & & & 0 & & 0 \\
 & & \uparrow{\scriptstyle \mathrm{d}^1} & & \uparrow{\scriptstyle \mathrm{d}^1} & & & & \uparrow{\scriptstyle \mathrm{d}^1} & & \uparrow \\
0 & \longrightarrow & \prod F_i^0 & \xrightarrow{\prod \mathrm{d}_i^0} & \prod F_i^1 & \xrightarrow{\prod \mathrm{d}_i^1} & \cdots & \longrightarrow & \prod F_i^p & \xrightarrow{\prod \mathrm{d}_i^p} & \prod F_i^{p+1} & \longrightarrow \cdots \\
 & & \uparrow{\scriptstyle \mathrm{d}^0} & & \uparrow{\scriptstyle \mathrm{d}^0} & & & & \uparrow{\scriptstyle \mathrm{d}^0} & & \uparrow \\
0 & \longrightarrow & \prod F_i^0 & \xrightarrow{\prod \mathrm{d}_i^0} & \prod F_i^1 & \xrightarrow{\prod \mathrm{d}_i^1} & \cdots & \longrightarrow & \prod F_i^p & \xrightarrow{\prod \mathrm{d}_i^p} & \prod F_i^{p+1} & \longrightarrow \cdots \\
 & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & & & 0 & & 0
\end{array} \tag{5.15}$$

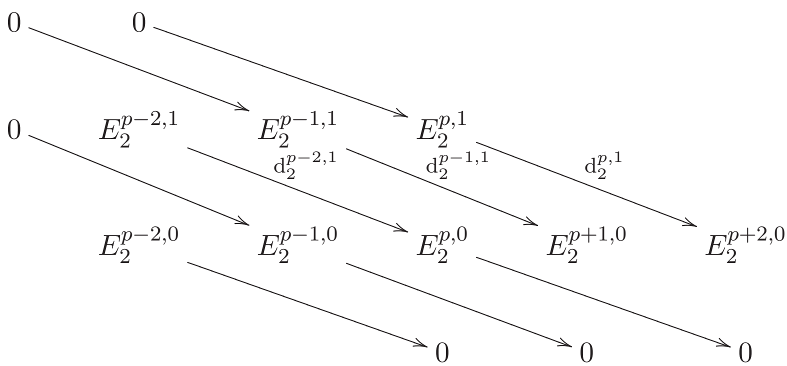
where

$$(\prod \mathrm{d}_i^p)((a_i^p)) = (\prod \mathrm{d}_i^p)(\dots, a_i^p, a_{i+1}^p, \dots) = \\ = (\dots, \mathrm{d}_i^p(a_i^p), \mathrm{d}_{i+1}^p(a_{i+1}^p), \dots) = \prod(\mathrm{d}_i^p(a_i^p)) \in \prod F_i^{p+1}.$$

Let $D^{\bullet,\bullet}$ be the double complex in (5.15), i.e., $D^{\bullet,\bullet} = (D^{p,q})_{p,q\geq 0}$, $D^{p,q} = 0$ unless $p \geq 0$ and $q = 0, 1$. Then from (3.9) and (3.24), we have the spectral sequences induced by two different filtrations

$$\begin{aligned} E_2^{p,q} &= \mathrm{H}^p_{\rightarrow}(\mathrm{H}^q_{\uparrow}(D^{\bullet,\bullet})) = \mathrm{H}^p(\varprojlim{}^{(q)} F_i^{\bullet}) \\ {}'E_2^{p,q} &= \mathrm{H}^p_{\uparrow}(\mathrm{H}^q_{\rightarrow}(D^{\bullet,\bullet})) = \varprojlim{}^{(p)}(\mathrm{H}^q(F_i^{\bullet})) \end{aligned} \tag{5.16}$$

abutting upon $E^n = \mathrm{H}^n(D^\bullet)$ where $D^n = \bigoplus_{p+q=n} D^{p,q}$ as in (3.2). Let us study $\{E_2^{p,1}\}$ and $\{E_2^{p,0}\}$ in detail. (Note $E_2^{p,q} = 0$ for $q \neq 0, 1$.) We have the following spectral sequence diagram with slope $-\frac{1}{2}$:

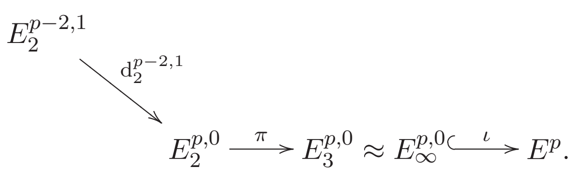


(5.17)

Then from (5.17), $E_3^{p,0}$ can be computed as

$$E_3^{p,0} = E_2^{p,0} / \operatorname{im} \mathrm{d}_2^{p-2,1},$$

and $E_3^{p,0} \approx E_4^{p,0} \approx E_\infty^{p,0}$. We get



(5.18)

From (5.17), the cohomology $E_3^{p-1,1}$ at $E_2^{p-1,1}$ is just $\ker \mathrm{d}_2^{p-1,1}$. Then we have $E_3^{p-1,1} \approx E_4^{p-1,1} \approx \cdots \approx E_\infty^{p-1,1}$. Since the abutment

$$E^{(p-1)+1} = E^p = E_\infty^{p-1,1} \bigoplus E_\infty^{p,0},$$

we get

$$E^p \xrightarrow{\pi_1} E_\infty^{p-1,1} \approx E_3^{p-1,1} = \ker \mathrm{d}_2^{p-1,1} \overset{\iota_1}{\hookrightarrow} E_2^{p-1,1} .$$

Consequently, we obtain

$$\begin{array}{ccccccccc}
\cdots \longrightarrow E_2^{p-1,1} & \xrightarrow{\mathrm{d}_2^{p-2,1}} & E_2^{p,0} & \xrightarrow{\iota\circ\pi} & E^p & \longrightarrow & E_2^{p-1,1} & \longrightarrow & \cdots \\
\| & & \| & & \| & & \| & & \\
\cdots \to \mathrm{H}^{p-2}(\varprojlim^{(1)} F_i^\bullet) & \longrightarrow & \mathrm{H}^p(\varprojlim F_i^\bullet) & \to & E^p & \to & \mathrm{H}^{p-1}(\varprojlim^{(1)} F_i^\bullet) & \to & \cdots
\end{array} \tag{5.19}$$

Next, we will study ${}'E_2^{p,q}$ of (5.16). The non-zero terms of ${}'E_2^{p,q}$ are the first two columns $\{{}'E_2^{0,q}\}$ and $\{{}'E_2^{1,q}\}$. From the spectral sequence $\{{}'E_2^{p,q}\}$ diagram like the one (5.15) for $\{E_2^{p,q}\}$, we have $0 = {}'E_2^{-1,q} \to {}'E_2^{1,q-1} \to {}'E_2^{3,q-2} = 0$ which implies

$${}'E_2^{1,q-1} \approx \cdots \approx {}'E_\infty^{1,q-1} \overset{\iota_1}{\hookrightarrow} E^q = {}'E_\infty^{1,q-1} \oplus {}'E_\infty^{0,q}.$$

Similarly, ${}'E_2^{0,q} \approx {}'E_\infty^{0,q} \xleftarrow{\pi_2} E^q$. That is, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}'E_2^{1,q-1} & \xrightarrow{\iota_2} & E^q & \xrightarrow{\pi_2} & {}'E_2^{0,q} & \longrightarrow & 0 \\
 & & \| & & \| & & \| & & \\
0 & \longrightarrow & \varprojlim^{(1)}(\mathrm{H}^{q-1}(F_i^\bullet)) & \longrightarrow & E^q & \longrightarrow & \varprojlim(\mathrm{H}^q(F_i^\bullet)) & \longrightarrow & 0.
\end{array} \tag{5.20}$$

In (5.19), if

$$E_2^{p-2,1} = \mathrm{H}^{p-2}(\varprojlim^{(1)} F_i^\bullet) = 0$$

and

$$E_2^{p-1,1} = \mathrm{H}^{p-1}(\varprojlim^{(1)} F_i^\bullet) = 0,$$

then we have the isomorphism from the abutment E^p to $E_2^{p,0} = \mathrm{H}^p(\varprojlim F_i^\bullet)$. In (5.20), if ${}'E_2^{1,q-1} = \varprojlim^{(1)}(\mathrm{H}^{q-1}(F_i^\bullet)) = 0$, then we get

$$E^q \xrightarrow{\approx} \varprojlim \mathrm{H}^q(F_i^\bullet) = {}'E_2^{0,q}.$$

Consequently, we would get the commutativity of H^* and $\varprojlim$, i.e.,

$$\mathrm{H}^p(\varprojlim F_i^\bullet \xrightarrow{\approx} \varprojlim \mathrm{H}^p(F_i^\bullet). \tag{5.21}$$

3.5.2 Vanishing of $\varprojlim^{(1)} F_i$

Let us recall: for $F \in \mathscr{C}^{\mathbb{Z}}$, i.e., an inverse system $(F_i)_{i\in\mathbb{Z}}$, the first cohomology

$$\mathrm{H}^1(C^\bullet) = \mathrm{R}^1\mathrm{H}^0(C^\bullet) \approx \mathrm{R}^1 \varprojlim F_i = \varprojlim^{(1)} F_i$$

of the complex

$$C^0 = \prod_{i\in\mathbb{Z}} F_i \xrightarrow{\mathrm{d}^0} C^1 = \prod_{i\in\mathbb{Z}} F_i \xrightarrow{\mathrm{d}^1} 0 \longrightarrow \cdots$$

is the cokernel $\operatorname{coker} \mathrm{d}^0 \approx \prod F_i / \operatorname{im} \mathrm{d}^0$. Recall also that $\mathrm{d}^0 : C^0 \to C^1$ is defined as $\mathrm{d}^0((a^i)) = F\phi(a^{i+1}) - a^i \in C^1 = \prod F_i$ for $(a^i) \in C^0 = \prod F_i$, where $i+1 \xrightarrow{\phi} i$ and $F_{i+1} \xrightarrow{F\phi} F_i$. We let $\phi_i^{i+1} = F\phi$ in what follows. For an arbitrary $(x_i) \in C^1$, we ask whether there is $(a_i) \in C^0$ to satisfy the following system $\mathrm{d}^0((a_i)) = (x_i)$ of equations:

$$\begin{cases} \phi_0^1(a_1) - a_0 = x_0, & \text{i.e.,} \quad \phi_0^1(a_1) = x_0 + a_0 \\ \phi_1^2(a_2) - a_1 = x_1, & \text{i.e.,} \quad \phi_1^2(a_2) = x_1 + a_1 \\ \quad\vdots & \quad\vdots \qquad\qquad \vdots \\ \phi_i^{i+1}(a_{i+1}) - a_i = x_i, & \text{i.e.,} \quad \phi_i^{i+1}(a_{i+1}) = x_i + a_i \\ \quad\vdots & \quad\vdots \qquad\qquad \vdots \end{cases} \tag{5.22}$$

When $\phi_i^{i+1} : F_{i+1} \to F_i$ is an epimorphism for $i = 0, 1, 2, \ldots$, it follows from (5.22) that one can find $(a_i) \in C^0$ satisfying $\mathrm{d}^0((a_i)) = (x_i) \in C^1$. Then $\mathrm{d}^0 : C^0 \to C^1$ is an epimorphism. That is, $\varprojlim^{(1)} F_i = \operatorname{coker} \mathrm{d}^0 = 0$.

More generally, for the inverse system $F = (F_i) \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}})$

$$\cdots \longrightarrow F_{i+1} \xrightarrow{\phi_i^{i+1}} F_i \xrightarrow{\phi_{i-1}^i} F_{i-1} \longrightarrow \cdots \xrightarrow{\phi_1^2} F_1 \xrightarrow{\phi_0^1} F_0 \tag{5.23}$$

when the sequence $\operatorname{im} \phi_0^1 \supset \operatorname{im} \phi_0^2 \supset \cdots$ becomes stationary, where

$$\phi_0^i := \phi_0^1 \circ \phi_1^2 \circ \cdots \circ \phi_{i-1}^i,$$

i.e., there exists $i_0 \in \mathbb{Z}$ to satisfy

$$\operatorname{im} \phi_0^{i_0} = \operatorname{im} \phi_0^j, \qquad \text{for all } j \geq i_0, \tag{5.24}$$

then $\mathrm{d}^0 : C^0 \to C^1$ is epimorphic. This is because for a given $(x_i) \in C^1$, one can choose $(a_i) \in C^0$ as follows:

$$\begin{cases} a_0 = -(x_0 + \phi_0^1(x_1) + \phi_0^2(x_2) + \cdots + \phi_0^{i_0}(x_{i_0})) \\ a_1 = -(x_1 + \phi_1^2(x_2) + \phi_1^3(x_3) + \cdots + \phi_1^{i_0}(x_{i_0})) \\ a_2 = -(x_2 + \phi_2^3(x_3) + \phi_2^4(x_4) + \cdots + \phi_2^{i_0}(x_{i_0})) \\ \qquad \vdots \\ a_{i_0-1} = -(x_{i_0-1} + \phi_{i_0-1}^{i_0}(x_{i_0})) \\ a_{i_0} = -x_{i_0} \\ a_i = 0, \quad \text{for} \quad i \geq i_0 + 1 \end{cases} \tag{5.25}$$

Then

$$\begin{aligned} \mathrm{d}^0((a_i)) &= (\phi_0^1(a_1) - a_0, \phi_1^2(a_2) - a_1, \phi_2^3(a_3) - a_2, \dots) = \\ &= \big(- \phi_0^1(x_1) - \phi_0^2(x_2) - \cdots - \phi_0^{i_0}(x_{i_0}) + \\ &\qquad + \phi_0^1(x_1) + \phi_0^2(x_2) + \cdots + \phi_0^{i_0}(x_{i_0}) + x_0, \dots, \\ &- \phi_{i_0-2}^{i_0-1}(x_{i_0-1}) - \phi_{i_0-2}^{i_0}(x_{i_0}) + \phi_{i_0-2}^{i_0-1}(x_{i_0-1}) + \phi_{i_0-2}^{i_0}(x_{i_0}) + x_{i_0-2}, \\ &- \phi_{i_0-1}^{i_0}(x_{i_0}) + \phi_{i_0}^{i_0}(x_{i_0}) + x_{i_0-1}, x_{i_0}, \dots \big) = \\ &= (x_0, x_1, \dots, x_{i_0-1}, x_{i_0}, x_{i_0+1}, \dots) = (x_i). \end{aligned}$$

Note that for the inverse system (F_i, ϕ_j^i) the condition in (5.24) is said to be the *Mittag-Leffler condition* for $(F_i) \in \mathrm{Ob}(\mathscr{C}^{\mathbb{Z}})$ at F_0. Furthermore, if $(\mathrm{H}^{q-1} F_i^{\bullet})$ satisfies the Mittag-Leffler condition, we have $\varprojlim^{(1)} \mathrm{H}^{q-1} F_i^{\bullet} = 0$. Then we obtain the isomorphism in (5.21).

Let us re-write the spectral sequences in (5.16) induced by the double complex (5.15) as

$$\begin{aligned} E_2^{p,q} &= \mathrm{H}^p(\varprojlim^{(q)} F_i^{\bullet}) = \mathrm{R}^p \mathrm{H}^0(\mathrm{R}^q \varprojlim F_i^{\bullet}) \\ {}'E_2^{p,q} &= \varprojlim^{(p)}(\mathrm{H}^q F_i^{\bullet}) = \mathrm{R}^p \varprojlim(\mathrm{R}^q \mathrm{H}^0(F_i^{\bullet})). \end{aligned} \tag{5.26}$$

We can consider

$$\begin{array}{ccc} \mathsf{Co}^+(\mathscr{C})^{\mathbb{Z}} & \xrightarrow{\ \varprojlim\ } & \mathsf{Co}^+(\mathscr{C}) \\ \Big\downarrow \mathrm{H}^0 & \searrow^{\mathsf{K}^0} & \Big\downarrow \mathrm{H}^0 \\ \mathscr{C}^{\mathbb{Z}} & \xrightarrow{\ \varprojlim\ } & \mathscr{C}\,. \end{array} \tag{5.27}$$

Then the spectral sequences in (5.26) abut upon the total cohomology

$$\mathrm{H}^n(D^\bullet) = \mathrm{R}^n\mathsf{K}^0((F_i^\bullet))$$

where $D^\bullet$ is the complex associated with the double complex (5.15). (See Section 3.3.) Note also that the cohomology functor H^0 commutes with the left exact functor $\varprojlim = \mathrm{H}^0_{C^\bullet}$, the cohomology of the complex constructed as $C^\bullet$ in (5.5).

Chapter 4

DERIVED CATEGORIES

4.1 Defining Derived Categories

4.1.1 Concepts Leading to Derived Categories

Let $\mathscr{A}$ be an abelian category and let $\mathsf{Co}^+(\mathscr{A})$ be the category of bounded from below complexes as before. Then as in Section 2.2 we can define the cohomology functor $\mathrm{H}^j : \mathsf{Co}^+(\mathscr{A}) \rightsquigarrow \mathscr{A}$ for $j \in \mathbb{Z}^+$. For a morphism

$$f^\bullet : (A^\bullet, \mathrm{d}_A^\bullet) \to (B^\bullet, \mathrm{d}_B^\bullet)$$

of complexes in $\mathsf{Co}^+(\mathscr{A})$ we have $\mathrm{H}^j(f^\bullet) : \mathrm{H}^j(A^\bullet) \to \mathrm{H}^j(B^\bullet)$ in $\mathscr{A}$. In Section 2.3 we found that for homotopic morphisms $f^\bullet$ and $g^\bullet$ from $A^\bullet$ to $B^\bullet$ their induced morphisms $\mathrm{H}^j(f^\bullet)$ and $\mathrm{H}^j(g^\bullet)$ are the same morphism from $\mathrm{H}^j(A^\bullet)$ to $\mathrm{H}^j(B^\bullet)$. That is, as the functor H^j from the homotopy category $\mathsf{K}^+(\mathscr{A}) = \mathsf{Co}^+(\mathscr{A})/\sim$ as defined in Section 2.3, $\mathrm{H}^j([f^\bullet])$ is independent of the choice of representative $f^\bullet$.

Next, let $\mathscr{A}$ and $\mathscr{B}$ be abelian categories and let $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ be an additive left exact functor. The question to ask is whether the assignment from $f^\bullet$ to $Ff^\bullet$ is a functor from $\mathsf{K}^+(\mathscr{A})$ to $\mathsf{K}^+(\mathscr{B})$ or not. The answer is positive: we need to prove the implication

$$f^\bullet \sim g^\bullet \Longrightarrow Ff^\bullet \sim Fg^\bullet.$$

For the additive functor F we get

$$F(f^j - g^j) = Ff^j - Fg^j = F('\mathrm{d}^{j-1}) \circ Fs^j + Fs^{j+1} \circ F\mathrm{d}^j,$$

where $s^j : A^j \to B^{j-1}$ are homotopy morphisms as in (3.2) in Chapter II. Namely, $Ff^\bullet$ is homotopic to $Fg^\bullet$.

A morphism $f^\bullet : A^\bullet \to {}'A^\bullet$of complexes is said to be a *quasi-isomorphism* when the induced morphism $\mathrm{H}^j(f^\bullet) : \mathrm{H}^j(A^\bullet) \to \mathrm{H}^j({}'A^\bullet)$ is an isomorphism in $\mathscr{A}$ for each j. Let us recall the diagram (3.33) in Subsection 3.3.6 in Chapter III:

$$\begin{array}{ccc} \mathsf{Co}^+(\mathscr{A}) & \overset{\mathsf{Co}^+F}{\rightsquigarrow} & \mathsf{Co}^+(\mathscr{B}) \\ \wr\downarrow \mathrm{H}^0 & \searrow^{\bar{F}} & \wr\downarrow \mathrm{H}^0 \\ \mathscr{A} & \overset{F}{\rightsquigarrow} & \mathscr{B}. \end{array} \tag{1.1}$$

The image of a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathsf{Co}^+(\mathscr{A})$ under the functor Co^+F in the above diagram is $Ff^\bullet : FA^\bullet \to FB^\bullet$ in $\mathsf{Co}^+(\mathscr{B})$, i.e.,

$$\begin{array}{ccc} A^\bullet & & FA^\bullet \\ \downarrow f^\bullet & \overset{\mathsf{Co}^+F}{\rightsquigarrow} & \downarrow Ff^\bullet \\ B^\bullet & & FB^\bullet \end{array} \tag{1.2}$$

Then we get the morphism between the associated spectral sequences as in (3.40) in Chapter III with hypercohomologies:

$$\begin{array}{ccc} E_1^{p,q}(A^\bullet) = \mathrm{R}^qFA^p & & E_2^{p,q}(A^\bullet) = \mathrm{H}^p(\cdots \to \mathrm{R}^qFA^p \to \cdots) \\ \downarrow & \text{or} & \downarrow \\ E_1^{p,q}(B^\bullet) = \mathrm{R}^qFB^p & & E_2^{p,q}(B^\bullet) = \mathrm{H}^p(\cdots \to \mathrm{R}^qFB^p \to \cdots) \end{array}$$

and the other one

$$\begin{array}{c} {}'E_2^{p,q}(A^\bullet) = \mathrm{R}^pF(\mathrm{H}^q(A^\bullet)) \\ \downarrow \\ {}'E_2^{p,q}(B^\bullet) = \mathrm{R}^pF(\mathrm{H}^q(B^\bullet)). \end{array} \tag{1.3}$$

We also have the morphism

$$E^n(A^\bullet) = \mathrm{R}^n\bar{F}A^\bullet \longrightarrow E^n(B^\bullet) = \mathrm{R}^n\bar{F}B^\bullet \tag{1.4}$$

between the abutments. Notice that for a quasi-isomorphism $f^\bullet : A^\bullet \to B^\bullet$, the morphism between ${}'E_2^{p,q}(A^\bullet)$ and ${}'E_2^{p,q}(B^\bullet)$ in (1.3) becomes an isomorphism. Then the morphism of the abutments in (1.4) is an isomorphism. Also note that the abutment $E^n(A^\bullet) = \mathrm{R}^n\bar{F}(A^\bullet)$ in (1.4) is not simply the cohomology $\mathrm{H}^n(FA^\bullet)$ of the complex $FA^\bullet$ as an object of $\mathsf{Co}^+(\mathscr{B})$. However, if

$E_1^{p,q}(A^\bullet) = \mathrm{R}^q F A^p = 0$ for $q \geq 1$ (i.e., A^p is an F-acyclic object, or A^p is an injective object), we would have

$$E_2^{n,0}(A^\bullet) = \mathrm{H}^n(E_1^{\bullet,0}) = \mathrm{H}^n(FA^\bullet) \approx E^n(A^\bullet),$$

the abutment. Namely, we are preparing to define a new category where a quasi-isomorphism is invertible (i.e, an isomorphism), and a homotopy equivalence class of morphisms matters. Such a category is said to be a derived category of $\mathscr{A}$.

4.1.2 Definition of Derived Category

Let $\mathscr{A}$ be an abelian category. We constructed $\mathsf{K}(\mathscr{A})$ from the abelian category $\mathsf{Co}(\mathscr{A})$ of complexes, where

$$\mathrm{Hom}_{\mathsf{K}(\mathscr{A})}(A^\bullet, B^\bullet) = \mathrm{Hom}_{\mathsf{Co}(\mathscr{A})}(A^\bullet, B^\bullet)\big/(\text{homotopy equivalence}). \quad (1.5)$$

The derived category $\mathsf{D}(\mathscr{A})$ is defined by the category obtained by localizing $\mathsf{K}(\mathscr{A})$ at the set (QIS) of quasi-isomorphisms:

$$\mathsf{D}(\mathscr{A}) := \mathsf{K}(\mathscr{A})_{(\mathrm{QIS})}. \quad (1.6)$$

Namely, there is a functor $\mathrm{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ such that $\mathrm{Q}_{\mathscr{A}}$ assigns quasi-isomorphisms in $\mathsf{K}(\mathscr{A})$ to isomorphisms in $\mathsf{D}(\mathscr{A})$. Then $\mathrm{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ satisfies the universal property as follows: if $F : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathscr{D}$ assigns quasi-isomorphisms of $\mathsf{K}(\mathscr{A})$ to isomorphisms of $\mathscr{D}$ then there is a unique functor $G : \mathsf{D}(\mathscr{A}) \rightsquigarrow \mathscr{D}$ satisfying the commutativity $F = G \circ \mathrm{Q}_{\mathscr{A}}$ in the diagram

$$\begin{array}{ccc} \mathsf{K}(\mathscr{A}) & \overset{F}{\rightsquigarrow} & \mathscr{D} \\ & {\scriptstyle \mathrm{Q}_{\mathscr{A}}}\searrow \quad \nearrow{\scriptstyle G} & \\ & \mathsf{D}(\mathscr{A}). & \end{array} \quad (1.7)$$

The functor $\mathrm{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ is said to be a *localizing functor*. The above $\mathsf{D}(\mathscr{A})$ can also be written as

$$\mathsf{D}(\mathscr{A}) := \mathsf{K}(\mathscr{A})[(\mathrm{QIS})^{-1}].$$

We will list properties that (QIS) satisfies:

QIS.1 If $s^\bullet$ and $t^\bullet$ are quasi-isomorphisms, the composition $s^\bullet \circ t^\bullet$ is also a quasi-isomorphism.

QIS.2 For a diagram

$$\begin{array}{ccc} & & B^\bullet \\ & & \downarrow {\scriptstyle s^\bullet} \\ A^\bullet & \xrightarrow{f^\bullet} & {}'B^\bullet \end{array}$$

in $\mathsf{K}(\mathscr{A})$, where $s^\bullet$ is a quasi-isomorphism, there exists a morphism ${}'f^\bullet$ and a quasi-isomorphism ${}'s^\bullet$ satisfying the commutativity $f^\bullet \circ {}'s^\bullet = s^\bullet \circ {}'f^\bullet$ of the diagram

$$\begin{array}{ccc} {}'A^\bullet & \xrightarrow{\ {}'f^\bullet\ } & B^\bullet \\ \downarrow{\scriptstyle {}'s^\bullet} & & \downarrow{\scriptstyle s^\bullet} \\ A^\bullet & \xrightarrow{\ f^\bullet\ } & {}'B^\bullet. \end{array}$$

QIS.3 For two morphisms $f^\bullet$ and $g^\bullet$ from $A^\bullet$ to ${}'B^\bullet$, the following (qis.3.1) and (qis.3.2) are equivalent:

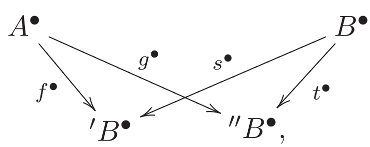
(qis.3.1) For a quasi-isomorphism $s^\bullet : {}'B^\bullet \to B^\bullet$, we have $s^\bullet \circ f^\bullet = s^\bullet \circ g^\bullet$.

(qis.3.2) For a quasi-isomorphism $t^\bullet : {}'A^\bullet \to A^\bullet$, we have $f^\bullet \circ t^\bullet = g^\bullet \circ t^\bullet$.

Since the derived category $\mathsf{D}(\mathscr{A})$ is the localized category of $\mathsf{K}(\mathscr{A})$ at (QIS), objects of $\mathsf{D}(\mathscr{A})$ are those of $\mathsf{K}(\mathscr{A})$ (hence of $\mathsf{Co}(\mathscr{A})$). Namely, an object of $\mathsf{D}(\mathscr{A})$ is a complex. On the other hand, a morphism ϕ from $A^\bullet$ to $B^\bullet$ in $\mathsf{D}(\mathscr{A})$ is an equivalence class of a pair $(f^\bullet, s^\bullet)$ of a morphism $f^\bullet$ and a quasi-isomorphism $s^\bullet$ given as in the diagram:

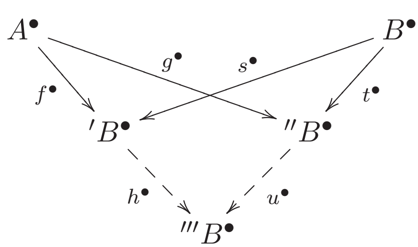
$$\begin{array}{ccc} A^\bullet & \xrightarrow{\ f^\bullet\ } & {}'B^\bullet \\ & \searrow{\scriptstyle \phi} & \uparrow{\scriptstyle s^\bullet} \\ & & B^\bullet \end{array} \tag{1.8}$$

for an object ${}'B^\bullet$. The equivalence relation between such pairs $(f^\bullet, s^\bullet)$ and $(g^\bullet, t^\bullet)$ is defined as follows. That is, for $(f^\bullet, s^\bullet)$ and $(g^\bullet, t^\bullet)$ given as



(1.9)

$(f^\bullet, s^\bullet)$ is equivalent to $(g^\bullet, t^\bullet)$, written as $(f^\bullet, s^\bullet) \sim (g^\bullet, t^\bullet)$, if and only if there are quasi-isomorphisms $h^\bullet : {}'B^\bullet \to {}'''B^\bullet$ and $u^\bullet : {}''B^\bullet \to {}'''B^\bullet$ satisfying the commutativity of the diagram



(1.10)

for an object $'''B^\bullet$ of $\mathsf{D}(\mathscr{A})$. When we write the localization of $\mathsf{K}(\mathscr{A})$ at (QIS) like the localization of the ring $\mathbb{Z}$ of integers at $(\mathbb{Z} - \{0\})$, we have

$$\frac{f^\bullet}{s^\bullet} = \frac{h^\bullet \circ f^\bullet}{h^\bullet \circ s^\bullet} = \frac{u^\bullet \circ g^\bullet}{u^\bullet \circ t^\bullet} = \frac{g^\bullet}{t^\bullet}. \tag{1.11}$$

That is, using the direct limit we have

$$\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(A^\bullet, B^\bullet) = \left\{ \varinjlim_{'B^\bullet} \left(\begin{array}{ccc} A^\bullet & & B^\bullet \\ & \searrow & \swarrow \text{q-i} \\ & {'B^\bullet} & \end{array} \right) \right\}, \tag{1.12}$$

where "q-i" means quasi-isomorphism. Let us write the equivalence class ϕ of $(f^\bullet, s^\bullet)$ by $f^\bullet/s^\bullet$. Then we will define the composition of morphisms in the derived category as follows. Let $\phi = f^\bullet/s^\bullet$ and $\psi = g^\bullet/t^\bullet$ be morphisms of $\mathsf{D}(\mathscr{A})$ given as

$$\begin{array}{ccccc} A^\bullet & \xrightarrow{f^\bullet} & {'B^\bullet} & \overset{'g^\bullet}{\dashrightarrow} & {''C^\bullet} \\ & \searrow^{\phi} & \uparrow{s^\bullet} & & \vdots\uparrow{'s^\bullet} \\ & & B^\bullet & \xrightarrow{g^\bullet} & {'C^\bullet} \\ & & & \searrow^{\psi} & \uparrow{t^\bullet} \\ & & & & C^\bullet. \end{array} \tag{1.13}$$

Then by QIS.2 in the above, there are $'g^\bullet : {'B^\bullet} \to {''C^\bullet}$ and $'s^\bullet : {'C^\bullet} \to {''C^\bullet}$ as shown in (1.13) where $'s^\bullet$ is a quasi-isomorphism. We define the composition $\psi \circ \phi : A^\bullet \to C^\bullet$ in $\mathsf{D}(\mathscr{A})$ by

$$\psi \circ \phi := ('g^\bullet \circ f^\bullet)/('s^\bullet \circ t^\bullet). \tag{1.14}$$

The reader may be interested in showing the independence of the choice of representatives $(f^\bullet, s^\bullet)$ and $(g^\bullet, t^\bullet)$ for the composition defined in (1.14), i.e., the well-definedness.

Next, we will define an addition $\boxplus$ in $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(A^\bullet, B^\bullet)$. Let $f^\bullet/s^\bullet$ and $'f^\bullet/'s^\bullet$ be elements of $\mathrm{Hom}_{\mathsf{D}(\mathscr{A})}(A^\bullet, B^\bullet)$. Then from

$$\begin{array}{ccc} A^\bullet & \xrightarrow{f^\bullet} & {'B^\bullet} \\ & & \uparrow{s^\bullet} \\ & & B^\bullet \end{array} \quad \boxplus \quad \begin{array}{ccc} A^\bullet & \xrightarrow{'f^\bullet} & {''B^\bullet} \\ & & \uparrow{'s^\bullet} \\ & & B^\bullet \end{array} \tag{1.15}$$

we extract

$$\begin{array}{ccc} {}'B^\bullet & \overset{r^\bullet}{\dashrightarrow} & {}'''B^\bullet \\ \Big\uparrow{\scriptstyle s^\bullet} & & \Big\uparrow{\scriptstyle {}'r^\bullet} \\ B^\bullet & \xrightarrow{{}'s^\bullet} & {}''B^\bullet. \end{array} \tag{1.16}$$

Then by QIS.2 we can complete the square in (1.16) by quasi-isomorphisms $r^\bullet$ and ${}'r^\bullet$. Define $f^\bullet/s^\bullet \boxplus {}'f^\bullet/{}'s^\bullet$ by

$$f^\bullet/s^\bullet \boxplus {}'f^\bullet/{}'s^\bullet := (r^\bullet \circ f^\bullet + {}'r^\bullet \circ {}'f^\bullet)/t^\bullet, \tag{1.17}$$

where $t^\bullet = r^\bullet \circ s^\bullet = {}'r^\bullet \circ {}'s^\bullet$. Namely, the addition in (1.15) equals

$$\begin{array}{ccc} A^\bullet & \xrightarrow{r^\bullet \circ f^\bullet + {}'r^\bullet \circ {}'f^\bullet} & {}'''B^\bullet \\ & & \Big\uparrow{\scriptstyle t^\bullet = r^\bullet \circ s^\bullet = {}'r^\bullet \circ {}'s^\bullet} \\ & & B^\bullet \end{array} \tag{1.18}$$

4.2 Derived Categorical Derived Functors

Let $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ be a additive left exact functor of abelian categories $\mathscr{A}$ and $\mathscr{B}$. Let $A^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{A}))$, i.e., $A^\bullet$ is a complex satisfying $A^j = 0$ for $j < 0$. Then consider a Cartan–Eilenberg resolution of $A^\bullet$ as in Subsection 3.3.3:

$$\begin{array}{ccccccccc} \vdots & & \vdots & & & & \vdots & & \\ \uparrow & & \uparrow & & & & \uparrow & & \\ I^{0,1} & \longrightarrow & I^{1,1} & \longrightarrow & \cdots & \longrightarrow & I^{p,1} & \longrightarrow & \cdots \\ \uparrow & & \uparrow & & & & \uparrow & & \\ I^{0,0} & \longrightarrow & I^{1,0} & \longrightarrow & \cdots & \longrightarrow & I^{p,0} & \longrightarrow & \cdots \\ \Big\uparrow{\scriptstyle \epsilon^0} & & \Big\uparrow{\scriptstyle \epsilon^1} & & & & \Big\uparrow{\scriptstyle \epsilon^p} & & \\ A^0 & \xrightarrow{\mathrm{d}^0} & A^1 & \xrightarrow{\mathrm{d}^1} & \cdots & \longrightarrow & A^p & \xrightarrow{\mathrm{d}^p} & \cdots \end{array} \tag{2.1}$$

Namely, all the $I^{p,q}$, $p, q \geq 0$ are injective objects of $\mathscr{A}$ satisfying $\mathrm{H}^q_\uparrow(I^{p,\bullet}) = 0$, $q \geq 1$ and $\mathrm{H}^0_\uparrow(I^{p,\bullet}) = A^p$ (i.e., $A^p \xrightarrow{\epsilon^p} I^{p,\bullet}$ is an injective resolution of A^p). We associate the spectral sequence

$$E_1^{p,q} = \mathrm{H}^q_\uparrow(I^{p,\bullet}) = 0, \qquad q \geq 1 \tag{2.2}$$

to the double complex $I^{\bullet,\bullet} = (I^{p,q})_{p,q\in\mathbb{Z}^+}$. (See Section 3.3.) We have the slope zero $E_1^{\bullet,0}$-term sequence

$$\begin{array}{ccccccc} E_1^{0,0} & \xrightarrow{d_1^{0,0}} & E_1^{1,0} & \xrightarrow{d_1^{1,0}} & E_1^{2,0} & \xrightarrow{d_1^{2,0}} & \cdots \\ \| & & \| & & \| & & \\ A^0 & \xrightarrow{d^0} & A^1 & \xrightarrow{d^1} & A^2 & \xrightarrow{d^2} & \cdots \end{array} \tag{2.3}$$

Therefore, $E_2^{p,0} = \mathrm{H}^p(E_1^{\bullet,0}) = \mathrm{H}^p(A^\bullet)$ holds. Since we have

$$0 = E_2^{p-2,1} \to E_2^{p,0} \to E_2^{p+2,-1} = 0,$$

we get $E_2^{p,0} \approx E_\infty^{p,0} \xrightarrow{\approx} E^p$. The abutment E^p is the total cohomology $\mathrm{H}^p(I^\bullet)$ of the single complex $I^\bullet$ where $I^n = \bigoplus_{p+q=n} I^{p,q}$ of the double complex $I^{\bullet,\bullet}$. That is, we obtain a complex $I^\bullet$ consisting of injective objects I^n, $n \geq 0$, which is quasi-isomorphic to $A^\bullet$, i.e.,

$$E_2^{p,0} = \mathrm{H}^p(A^\bullet) \xrightarrow{\approx} \mathrm{H}^p(I^\bullet) = E^p.$$

For an additive left exact functor F from $\mathscr{A}$ to $\mathscr{B}$ we can give the definition of the derived functor $\mathbb{R}F$ from $\mathsf{D}^+(\mathscr{A})$ to $\mathsf{D}^+(\mathscr{B})$ as we did in Section 2.7 in Chapter II as follows. Let $I^\bullet$ be an injective complex which is quasi-isomorphic to $A^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{A}))$. Define the derived functor $\mathbb{R}FA^\bullet$ of $A^\bullet$ by $FI^\bullet \in \mathrm{Ob}(\mathsf{D}^+(\mathscr{A}))$, i.e.,

$$\mathbb{R}FA^\bullet := FI^\bullet. \tag{2.4}$$

Define also

$$\mathbb{R}^j FA^\bullet := \mathrm{H}^j(FI^\bullet). \tag{2.5}$$

Note that for an injective object I^p we have $E_1^{p,q} = \mathrm{R}^q F I^p = 0, q \geq 1$. Then the abutment $\mathrm{R}^j \bar{F} A^\bullet$, the hyperderived functor, is isomorphic to the cohomology $\mathrm{H}^j(E_1^{\bullet,0}) = \mathrm{H}^j(\mathrm{R}^0 FI^\bullet) = \mathrm{H}^j(FI^\bullet)$. Therefore, the right hand-side of (2.5) is isomorphic to the hyperderived functor, i.e.,

$$\mathbb{R}^j FA^\bullet \xrightarrow{\approx} \mathrm{R}^j \bar{F} A^\bullet. \tag{2.6}$$

Let us observe that for the additive left exact functor $F : \mathscr{A} \rightsquigarrow \mathscr{B}$, a quasi-isomorphism $A^\bullet \xrightarrow{s^\bullet} I^\bullet$ is assigned to $FA^\bullet \xrightarrow{Fs^\bullet} FI^\bullet$. By taking the cohomology the induced morphism becomes $\mathrm{H}^j(FA^\bullet) \xrightarrow{\mathrm{H}^j(Fs^\bullet)} \mathrm{H}^j(FI^\bullet)$.

Namely, for the quasi-isomorphism $A^\bullet \xrightarrow{s^\bullet} I^\bullet$ we have the following diagram

$$
\begin{array}{ccccc}
FA^\bullet & \xrightarrow{\quad Fs^\bullet \quad} & \mathbb{R}FA^\bullet := FI^\bullet & & \\
 & \Big\downarrow{\scriptstyle \mathrm{H}^j} & & & \\
\mathrm{H}^j(FA^\bullet) & \xrightarrow[\mathrm{H}^j(Fs^\bullet)]{} & \mathbb{R}^jFA^\bullet & \xrightarrow{\ \approx\ } & \mathrm{R}^j\bar{F}A^\bullet \\
 & & & & \Big\downarrow{\scriptstyle \approx} \\
 & & & & \mathrm{H}^j(FI^\bullet).
\end{array}
\tag{2.7}
$$

Let us re-write the above diagram for the case of an injective resolution $A \xrightarrow{\epsilon} I^\bullet$ of a single object $A \in \mathrm{Ob}(\mathscr{A})$:

$$
\begin{array}{ccccc}
FA & \xrightarrow{\quad F\epsilon \quad} & \mathbb{R}FA := FI^\bullet & & \\
 & \Big\downarrow{\scriptstyle \mathrm{H}^j} & & & \\
\mathrm{H}^j(FA) & \xrightarrow[\mathrm{H}^j(F\epsilon)]{} & \mathrm{H}^j(FI^\bullet) : & = & \mathrm{R}^jFA \\
 & & & & \Big\| \\
 & & & & \mathbb{R}^jFA.
\end{array}
\tag{2.8}
$$

The right hand-side of the lower part of the above diagram (2.8) indicates that the derived functor, as defined in Chapter II, coincides with the notion of the derived functor in the sense of the derived category. Since the left hand-side of the lower part of (2.8) is the cohomology of the single object FA we have $\mathrm{H}^j(FA) = 0$ for $j \geq 1$ and $\mathrm{H}^0(FA) \approx FA$.

Next we will confirm that two quasi-isomorphisms $s^\bullet$ and $r^\bullet$ from $A^\bullet$ to two injective complexes

$$
\begin{array}{ccc}
I^\bullet & \overset{'r^\bullet}{\dashrightarrow} & K^\bullet \\
{\scriptstyle s^\bullet}\Big\uparrow & & \Big\uparrow{\scriptstyle 's^\bullet} \\
A^\bullet & \xrightarrow{\ r^\bullet\ } & J^\bullet
\end{array}
\tag{2.9}
$$

provide the isomorphic objects $FI^\bullet$ and $FJ^\bullet$ in $\mathsf{D}^+(\mathscr{B})$. By QIS.2 in Subsection 4.1.2 (whose proof is not given here), we can complete the square as in (2.9) by quasi-isomorphisms $'s^\bullet$ and $'r^\bullet$ where the complex $K^\bullet$ is the direct sum of $I^\bullet$ and $J^\bullet$. Then the functor F takes the commutative diagram (2.9) to the following commutative diagram

$$\begin{array}{ccc} FI^\bullet & \xrightarrow{F'r^\bullet} & FK^\bullet \\ {\scriptstyle Fs^\bullet}\uparrow & & \uparrow{\scriptstyle F's^\bullet} \\ FA^\bullet & \xrightarrow{Fr^\bullet} & FJ^\bullet \end{array} \tag{2.10}$$

whose cohomologies are

$$\begin{array}{ccc} \mathrm{H}^j(FI^\bullet) & \xrightarrow{\mathrm{H}^j(F'r^\bullet)} & \mathrm{H}^j(FK^\bullet) \\ {\scriptstyle \mathrm{H}^j(Fs^\bullet)}\uparrow & & \uparrow{\scriptstyle \mathrm{H}^j(F's^\bullet)} \\ \mathrm{H}^j(FA^\bullet) & \xrightarrow{\mathrm{H}^j(Fr^\bullet)} & \mathrm{H}^j(FJ^\bullet). \end{array} \tag{2.11}$$

The first row of (2.11) is $\mathrm{H}^j(FI^\bullet) = \mathrm{R}^j\bar{F}I^\bullet \to \mathrm{H}^j(FK^\bullet) = \mathrm{R}^j\bar{F}K^\bullet$ (as shown in Subsection 4.1.2) since $E_1^{p,q}(I^\bullet) = \mathrm{R}^qFI^p = 0$ and

$$E_1^{p,q}(K^\bullet) = \mathrm{R}^qFK^p = 0$$

for $q \geq 1$. On the other hand, for the quasi-isomorphism $'r^\bullet : I^\bullet \to K^\bullet$, the isomorphism between the $'E_2^{p,q}$-terms

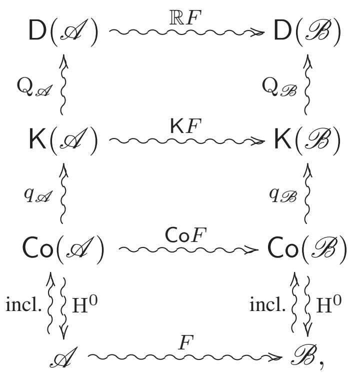
$$'E_2^{p,q}(I^\bullet) = \mathrm{R}^pF(\mathrm{H}^q(I^\bullet)) \xrightarrow{\approx} {'E_2^{p,q}}(K^\bullet) = \mathrm{R}^pF(\mathrm{H}^q(K^\bullet))$$

induces the isomorphism between abutments

$$E^j(I^\bullet) = \mathrm{R}^j\bar{F}I^\bullet$$

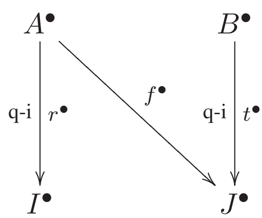
and $E^j(K^\bullet) = \mathrm{R}^j\bar{F}K^\bullet$. Consequently the morphism $F'r^\bullet : FI^\bullet \to FK^\bullet$ (and similarly, $F's^\bullet : FJ^\bullet \to FK^\bullet$) is a quasi-isomorphism. (See Subsection 4.1.1 for the above argument.) Therefore, we get the isomorphism between $\mathrm{H}^j(FI^\bullet)$ and $\mathrm{H}^j(FJ^\bullet)$. Hence, the definition of $\mathbb{R}FA^\bullet$ in (2.4) is independent of the choice of quasi-isomorphism from $A^\bullet$ to an injective complex $I^\bullet$. Note also that $\mathrm{H}^j(Fs^\bullet)$ and $\mathrm{H}^j(Fr^\bullet)$ need not be isomorphisms for $j \geq 1$.

The process from $\mathsf{Co}(\mathscr{A})$ to $\mathsf{D}(\mathscr{A})$ through $\mathsf{K}(\mathscr{A})$ is and from $F : \mathscr{A} \rightsquigarrow \mathscr{B}$ to $\mathbb{R}F : \mathsf{D}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ are summarized in the following diagram:



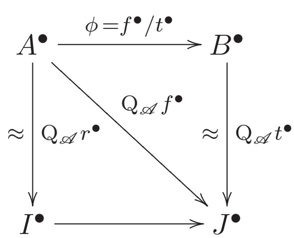
(2.12)

where $q_{\mathscr{A}}$ is defined by (1.5). Namely, for $g \circ f \sim 1$ we have $[g \circ f] = [1]$, i.e., $[g] \circ [f] = 1$ in $\mathsf{K}(\mathscr{A})$. The functor $\mathrm{Q}_{\mathscr{A}}$ assign a quasi-isomorphism in $\mathsf{K}(\mathscr{A})$ to an isomorphism in $\mathsf{D}(\mathscr{A})$. That is, for objects $A^\bullet$ and $B^\bullet$ in $\mathsf{K}(\mathscr{A})$ consider



(2.13)

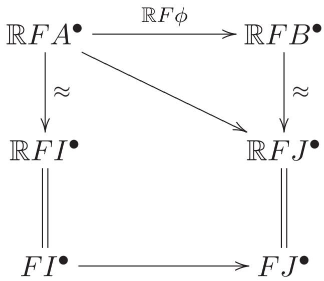
in $\mathsf{K}(\mathscr{A})$ where q-i is a quasi-isomorphism, and $I^\bullet$ and $J^\bullet$ are injective complexes as in Section 4.2. Then the functor $\mathrm{Q}_{\mathscr{A}}$ assigns the morphism to



(2.14)

in $\mathsf{D}(\mathscr{A})$ where all the quasi-isomorphisms become isomorphism. Note that in $\mathsf{D}(\mathscr{A})$ we can have the morphism $\mathrm{Q}_{\mathscr{A}} f^\bullet \circ (\mathrm{Q}_{\mathscr{A}} r^\bullet)^{-1} : I^\bullet \to J^\bullet$. Finally

$\mathbb{R}F : \mathsf{D}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{B})$ takes the diagram (2.14) to

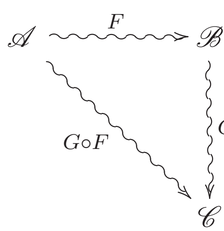


(2.15)

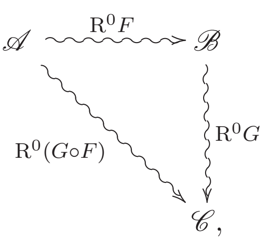
where in $\mathsf{D}(\mathscr{B})$ we can define $\mathbb{R}F\phi : \mathbb{R}FA^\bullet \to \mathbb{R}FB^\bullet$ by

$$F(\mathrm{Q}_{\mathscr{A}} f^\bullet \circ (\mathrm{Q}_{\mathscr{A}} t^\bullet)^{-1}) = F(\mathrm{Q}_{\mathscr{A}} f^\bullet) \circ F((\mathrm{Q}_{\mathscr{A}} t^\bullet)^{-1}) = Ff^\bullet/Ft^\bullet.$$

As an application of the concept of a derived category, we consider the case of a composite functor as in Subsection 3.3.2. Namely, let F be a left exact additive functor of abelian categories $\mathscr{A}$ and $\mathscr{B}$ with enough injectives and let $G : \mathscr{B} \rightsquigarrow \mathscr{C}$ also be a left exact functor to the abelian category $\mathscr{C}$. Furthermore, assume that the image object FI in $\mathscr{B}$ of an injective object I of $\mathscr{A}$ is G-acyclic, i.e., $\mathrm{R}^j G(FI) = 0$ for $j \geq 1$. As in Subsection 3.3.2, for the diagram



we have the commutative diagram for the 0-th derived functors



(2.16)

i.e., $\mathrm{R}^0(G \circ F) \approx G \circ F \approx \mathrm{R}^0 G \circ \mathrm{R}^0 F$. The concept of a derived category enables this commutativity even for higher cohomologies. That is, we have the commutative diagram of the derived categories associated with the above

(2.16):

$$\begin{array}{ccc} \mathsf{D}^+(\mathscr{A}) & \overset{\mathbb{R}F}{\rightsquigarrow} & \mathsf{D}^+(\mathscr{B}) \\ & \searrow^{\mathbb{R}(G\circ F)} & \downarrow{\scriptstyle \mathbb{R}G} \\ & & \mathsf{D}^+(\mathscr{C}), \end{array} \tag{2.17}$$

i.e, we have

$$\mathbb{R}(G \circ F) = \mathbb{R}G \circ \mathbb{R}F. \tag{2.18}$$

We will prove (2.18) as follows. For an object $A^\bullet$ of $\mathsf{D}^+(\mathscr{A})$, let $I^\bullet$ be an injective complex which is quasi-isomorphic to $A^\bullet$. By the definition of the derived functor $\mathbb{R}FA^\bullet$ of F, we have $\mathbb{R}FA^\bullet = FI^\bullet$. Next we will compute $\mathbb{R}G(FI^\bullet)$ as follows. The cohomology $\mathbb{R}^jG(FI^\bullet)$ of the complex $\mathbb{R}G(FI^\bullet)$ in $\mathsf{D}^+(\mathscr{C})$ is the hyperderived functor $\mathrm{R}^j\bar{G}(FI^\bullet)$. One of the spectral sequences having the hyperderived functor $\mathrm{R}^j\bar{G}(FI^\bullet)$ as the abutment is

$$E_1^{p,q} = \mathrm{R}^qG(FI^p) \tag{2.19}$$

as in (3.43). By the G-acyclicity assumption on FI^p we have $E_1^{p,q} = 0$ for $q \geq 1$ where $E_1^{p,0} = \mathrm{R}^0G(FI^p) \approx G(FI^p)$. The spectral sequence of $E_1^{p,q}$-terms with slope zero becomes

$$\begin{array}{ccccccccc} \cdots \longrightarrow & E_1^{p-1,0} & \longrightarrow & E_1^{p,0} & \longrightarrow & E_1^{p+1,0} & \longrightarrow \cdots \\ & \| & & \| & & \| & \\ \cdots \to & (G\circ F)I^{p-1} & \to & (G\circ F)I^p & \to & (G\circ F)I^{p+1} & \to \cdots . \end{array} \tag{2.20}$$

Then $E_2^{p,q} = \mathrm{H}^p((G \circ F)I^\bullet)$ satisfying $E_2^{p,0} \approx E_\infty^{p,0} \approx E^p = \mathrm{R}^p\bar{G}(FI^\bullet)$. Summarizing the above we get

$$\begin{aligned} E_2^{p,0} &= \mathbb{R}^p(G \circ F)A^\bullet := \\ &:= \mathrm{H}^p((G \circ F)I^\bullet) \approx \\ &\approx E^p = \mathrm{R}^p\bar{G}(FI^\bullet) = \\ &= \mathbb{R}^pG(FI^\bullet). \end{aligned}$$

Namely, the two complexes $\mathbb{R}(G \circ F)A^\bullet$ and $\mathbb{R}G(FI^\bullet) = \mathbb{R}G(\mathbb{R}FA^\bullet)$ are quasi-isomorphic. Therefore, as objects in $\mathsf{D}^+(\mathscr{C})$ we have

$$\mathbb{R}(G \circ F)A^\bullet \approx \mathbb{R}G(\mathbb{R}FA^\bullet).$$

4.3 Triangles

Let $\mathscr{A}$ be an abelian category. We will define an autofunctor $[n]$ on the category $\mathsf{Co}(\mathscr{A})$ of complexes as follows:

$$[n] : \mathsf{Co}(\mathscr{A}) \rightsquigarrow \mathsf{Co}(\mathscr{A}) \tag{3.1}$$

is defined by $[n]A^\bullet := A^{\bullet+n}$ and $[n]\mathrm{d}^\bullet_{A^\bullet} := (-1)^n \mathrm{d}^{\bullet+n}_{A^\bullet}$ for

$$(A^\bullet, \mathrm{d}^\bullet_{A^\bullet}) \in \mathrm{Ob}(\mathsf{Co}(\mathscr{A})).$$

We usually write $A[n]^\bullet$ and $\mathrm{d}_{A^\bullet}[n]^\bullet$ for $[n]A^\bullet$ and $[n]\mathrm{d}^\bullet_{A^\bullet}$ respectively. Namely, $A[n]^j = A^{j+n}$ and $\mathrm{d}_{A^\bullet}[n]^j = (-1)^n \mathrm{d}^{j+n}_{A^\bullet}$. For example, for $[1] : \mathsf{Co}(\mathscr{A}) \rightsquigarrow \mathsf{Co}(\mathscr{A})$ a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathsf{Co}(\mathscr{A})$, i.e., more explicitly

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A^j & \xrightarrow{\mathrm{d}^j_{A^\bullet}} & A^{j+1} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{j+1}} & \\
\cdots \longrightarrow & B^j & \xrightarrow{\mathrm{d}^j_{B^\bullet}} & B^{j+1} & \longrightarrow \cdots ,
\end{array}$$

$[1]f^\bullet = f[1]^\bullet : A[1]^\bullet \to B[1]^\bullet$ becomes

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A^{j+1} & \xrightarrow{-\mathrm{d}^{j+1}_{A^\bullet}} & A^{j+2} & \longrightarrow \cdots \\
& \downarrow{\scriptstyle f^{j+1}} & & \downarrow{\scriptstyle f^{j+2}} & \\
\cdots \longrightarrow & B^{j+1} & \xrightarrow{-\mathrm{d}^{j+1}_{B^\bullet}} & B^{j+2} & \longrightarrow \cdots .
\end{array}$$

Let $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to C^\bullet$ be two morphisms of complexes in $\mathsf{K}(\mathscr{A})$. Then we have $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet$. When there is a morphism $h^\bullet : C^\bullet \to A[1]^\bullet$ in $\mathsf{K}(\mathscr{A})$,

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet \tag{3.2}$$

is said to be a *triangle* in $\mathsf{K}(\mathscr{A})$. We sometimes write such a triangle (3.2) as

$$\begin{array}{ccc}
A^\bullet & \xleftarrow[{[1]}]{h^\bullet} & C^\bullet \\
& {\scriptstyle f^\bullet}\searrow \quad \nearrow{\scriptstyle g^\bullet} & \\
& B^\bullet &
\end{array} \tag{3.3}$$

A morphism of triangles is $(\alpha^\bullet, \beta^\bullet, \gamma^\bullet, \alpha[1]^\bullet)$ of the commutative diagram of the top and bottom triangles of

$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{g^\bullet} & C^\bullet & \xrightarrow{h^\bullet} & A[1]^\bullet \\
\downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle \beta^\bullet} & & \downarrow{\scriptstyle \gamma^\bullet} & & \downarrow{\scriptstyle \alpha[1]^\bullet} \\
{}'A^\bullet & \xrightarrow{'f^\bullet} & {}'B^\bullet & \xrightarrow{'g^\bullet} & {}'C^\bullet & \xrightarrow{'h^\bullet} & {}'A[1]^\bullet
\end{array} \tag{3.4}$$

in $\mathsf{K}(\mathscr{A})$. When $\alpha^\bullet$, $\beta^\bullet$ and $\gamma^\bullet$ are isomorphisms of $\mathsf{K}(\mathscr{A})$, triangles

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow A[1]^\bullet$$

and

$$'A^\bullet \longrightarrow {}'B^\bullet \longrightarrow {}'C^\bullet \longrightarrow {}'A[1]^\bullet$$

are said to be *isomorphic triangles*.

For an arbitrarily given morphism $f^\bullet : A^\bullet \to B^\bullet$ of complexes, we can construct a complex $C^\bullet_{f^\bullet}$ and morphisms $b^\bullet$ and $a^\bullet$ so that

$$A^\bullet \longrightarrow B^\bullet \xrightarrow{b^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet \tag{3.5}$$

may become a triangle. Define the complex $C^\bullet_{f^\bullet}$ by

$$C^j_{f^\bullet} := A^{j+1} \oplus B^j = A[1]^j \oplus B^j \tag{3.6}$$

and $\mathrm{d}^j_{C^\bullet} : C^j_{f^\bullet} \to C^{j+1}_{f^\bullet}$ by

$$\begin{aligned} \mathrm{d}^j_{C^\bullet}\begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix} &= \begin{bmatrix} \mathrm{d}_{A^\bullet}[1]^j & 0 \\ f[1]^j & \mathrm{d}^j_{B^\bullet} \end{bmatrix}\begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix} = \\ &= \begin{pmatrix} -\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) \\ f^{j+1}(x^{j+1}) + \mathrm{d}^j_{B^\bullet}(y^j) \end{pmatrix} \in C^{j+1}_{f^\bullet} \end{aligned} \tag{3.7}$$

Then we have

$$\begin{aligned} &\mathrm{d}^{j+1}_{C^\bullet} \circ \mathrm{d}^j_{C^\bullet}\left(\begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix}\right) = \\ &\quad = \begin{pmatrix} \mathrm{d}^{j+2}_{A^\bullet} \circ \mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) \\ f^{j+2}(-\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1})) + \mathrm{d}^{j+1}_{B^\bullet}(f^{j+1}(x^{j+1}) + \mathrm{d}^j_{B^\bullet}(y^j)) \end{pmatrix} = \\ &\quad = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \in C^{j+2}_{f^\bullet} \end{aligned}$$

from the commutativity of the diagram

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & A^j & \xrightarrow{\mathrm{d}^j_{A^\bullet}} & A^{j+1} & \xrightarrow{\mathrm{d}^{j+1}_{A^\bullet}} & A^{j+2} & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle f^j} & & \downarrow{\scriptstyle f^{j+1}} & & \downarrow{\scriptstyle f^{j+2}} & & \\ \cdots & \longrightarrow & B^j & \xrightarrow{\mathrm{d}^j_{B^\bullet}} & B^{j+1} & \xrightarrow{\mathrm{d}^{j+1}_{B^\bullet}} & B^{j+2} & \longrightarrow & \cdots . \end{array} \tag{3.8}$$

Namely, $(C^\bullet_{f^\bullet}, \mathrm{d}^\bullet_{C^\bullet})$ is a complex. For $C^\bullet_{f^\bullet} = A[1]^\bullet \oplus B^\bullet$ define $b^\bullet : B^\bullet \to C^\bullet_{f^\bullet}$ and $a^\bullet : C^\bullet_{f^\bullet} \to A[1]^\bullet$ in (3.5) by $b^\bullet := \left[\begin{smallmatrix} 0^\bullet \\ 1^\bullet_{B^\bullet} \end{smallmatrix}\right]$ and $a^\bullet := [1_{A^\bullet}[1]^\bullet, 0^\bullet]$. Then

(3.5) becomes a triangle. Notice that $0 \to B^\bullet \to C^\bullet_{f^\bullet} \to A[1]^\bullet \to 0$ is an exact sequence in $\mathsf{Co}(\mathscr{A})$. A triangle $A^\bullet \to B^\bullet \to C^\bullet \to A[1]^\bullet$ is said to be a *distinguished triangle* when for a morphism $'A^\bullet \xrightarrow{'f^\bullet} {'B^\bullet}$ of complexes there is an isomorphism of triangles

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{g^\bullet} & C^\bullet & \xrightarrow{h^\bullet} & A[1]^\bullet \\ \downarrow\approx & & \downarrow\approx & & \downarrow\approx & & \downarrow\approx \\ 'A^\bullet & \xrightarrow{'f^\bullet} & 'B^\bullet & \xrightarrow{'b^\bullet} & C^\bullet_{'f^\bullet} & \xrightarrow{'a^\bullet} & 'A[1]^\bullet \end{array} \tag{3.9}$$

in $\mathsf{K}(\mathscr{A})$.

The complex $C^\bullet_{f^\bullet}$ in (3.5) is said to be the *mapping cone* of $f^\bullet : A^\bullet \to B^\bullet$. Notice that $C^\bullet_{f^\bullet}$ depends upon a homotopy equivalence class. Namely, if we have $f_1^\bullet \sim f_2^\bullet$ then there is an isomorphism $C^\bullet_{f_1^\bullet} \approx C^\bullet_{f_2^\bullet}$ in $\mathsf{K}(\mathscr{A})$. For $f^\bullet : A^\bullet \to B^\bullet$, we will define another complex $'C^\bullet_{f^\bullet}$ so that the associated triangles

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{b^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet} & A[1]^\bullet \\ \| & & \downarrow{'b^\bullet} & & \| & & \| \\ A^\bullet & \xrightarrow{\iota^\bullet} & 'C^\bullet_{f^\bullet} & \xrightarrow{\pi^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet} & A[1]^\bullet \end{array} \tag{3.10}$$

become isomorphic triangles. Define the complex by

$$'C^\bullet_{f^\bullet} := A^\bullet \oplus A[1]^\bullet \oplus B^\bullet, \tag{3.11}$$

where $\mathrm{d}^\bullet_{'C^\bullet} : {'C^j_{f^\bullet}} \to {'C^{j+1}_{f^\bullet}}$ is defined as

$$\begin{bmatrix} \mathrm{d}^j_{A^\bullet} & -1^{j+1} & 0 \\ 0 & \mathrm{d}_{A^\bullet}[1]^j & 0 \\ 0 & f[1]^\bullet & \mathrm{d}^j_{B^\bullet} \end{bmatrix} \begin{pmatrix} x^j \\ x^{j+1} \\ y^j \end{pmatrix} = \begin{pmatrix} \mathrm{d}^j_{A^\bullet}(x^j) - x^{j+1} \\ -\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) \\ f^{j+1}(x^{j+1}) - \mathrm{d}^j_{B^\bullet}(y^j) \end{pmatrix}. \tag{3.12}$$

Then by the commutative diagram (3.8) we have $\mathrm{d}^{j+1}_{'C^\bullet} \circ \mathrm{d}^j_{'C^\bullet} = 0$ obtaining the complex $'C^\bullet_{f^\bullet}$ which is said to be the *mapping cylinder* of $f^\bullet : A^\bullet \to B^\bullet$. Morphisms $\iota^\bullet$, $\pi^\bullet$ and $'b^\bullet$ in (3.11) are given by

$$\iota^\bullet := \begin{bmatrix} 1^\bullet \\ 0 \\ 0 \end{bmatrix}, \quad \pi^\bullet := \begin{bmatrix} 0 & 1^\bullet & 0 \\ 0 & 0 & 1^\bullet \end{bmatrix}, \quad \text{and} \quad 'b^\bullet = \begin{bmatrix} 0 \\ 0 \\ 1^\bullet \end{bmatrix}. \tag{3.13}$$

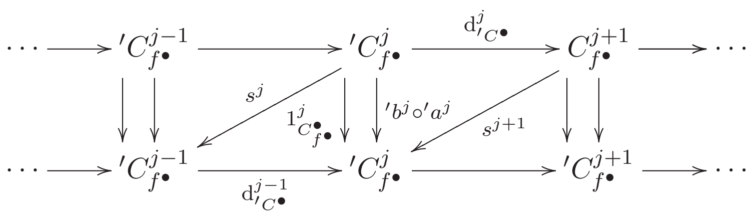
Define $'a^\bullet : {'C^\bullet_{f^\bullet}} \to B^\bullet$ by

$$'a^j \begin{pmatrix} x^j \\ x^{j+1} \\ y^j \end{pmatrix} := f^j(x^j) + y^j.$$

Then we have ${}'a^\bullet \circ {}'b^\bullet = 1_{B^\bullet}$. Notice, however, that ${}'b^\bullet \circ {}'a^\bullet$ is not $1_{'C^\bullet_{f^\bullet}}$, but homotopic to $1_{C^\bullet_{f^\bullet}}$, i.e., ${}'b^\bullet \circ {}'a^\bullet \sim 1_{C^\bullet_{f^\bullet}}$. That is, by defining

$$s^\bullet : {}'C^\bullet_{f^\bullet} \to {}'C^\bullet_{f^\bullet}[-1]$$

in the diagram



as $s^j\begin{pmatrix} x^j \\ x^{j+1} \\ y^j \end{pmatrix} = \begin{pmatrix} 0 \\ x^j \\ 0 \end{pmatrix}$, we get

$$1^j_{'C^\bullet_{f^\bullet}} - {}'b^j \circ {}'a^k = s^{j+1} \circ \mathrm{d}^j_{'C^\bullet} + \mathrm{d}^{j-1}_{'C^\bullet} \circ s^j,$$

i.e., ${}'b^j \circ {}'a^j$ is homotopic to $1_{'C^\bullet_{f^\bullet}}$. The cohomology

$$\mathrm{H}^j({}'b^\bullet \circ {}'a^\bullet) = \mathrm{H}^j({}'b^\bullet) \circ \mathrm{H}^j({}'a^\bullet) = \mathrm{H}^j(1_{'C^\bullet_{f^\bullet}}) = 1_{\mathrm{H}^j('C^\bullet_{f^\bullet})}$$

implies that the quasi-isomorphisms ${}'a^\bullet$ and ${}'b^\bullet$ are isomorphisms in $\mathsf{D}(\mathscr{A})$.

Summarizing the above computation: for a morphism $f^\bullet : A^\bullet \to B^\bullet$ in $\mathsf{Co}(\mathscr{A})$ we have the commutative diagram

$$\begin{CD}
@. 0 @>>> B^\bullet @>b^\bullet>> C^\bullet_{f^\bullet} @>a^\bullet>> A[1]^\bullet @>>> 0\\
@. @. @V{'b^\bullet}VV @| \\
0 @>>> A^\bullet @>\iota^\bullet>> {}'C^\bullet_{f^\bullet} @>\pi^\bullet>> C^\bullet_{f^\bullet} @>>> 0\\
@. @| @V{'a^\bullet}VV \\
@. A^\bullet @>>> B^\bullet
\end{CD} \tag{3.14}$$

satisfying ${}'a^\bullet \circ {}'b^\bullet = 1_{B^\bullet}$ and ${}'b^\bullet \circ {}'a^\bullet \sim 1_{C^\bullet_{f^\bullet}}$. In $\mathsf{D}(\mathscr{A})$, ${}'a^\bullet$ and ${}'b^\bullet$ are isomorphisms between $B^\bullet$ and ${}'C^\bullet_{f^\bullet}$.

4.4 Triangles for Exact Sequences

For a short exact sequence

$$0 \longrightarrow {}'A^\bullet \longrightarrow A^\bullet \longrightarrow {}''A^\bullet \longrightarrow 0 \tag{4.1}$$

in the category $\mathsf{Co}^+(\mathscr{A})$ of complexes of an abelian category $\mathscr{A}$, through the connecting morphisms $\partial^j : \mathrm{H}^j({}''A^\bullet) \to \mathrm{H}^{j+1}({}'A^\bullet)$, $j \geq 0$, we get the long exact sequence on cohomology

$$0 \longrightarrow \mathrm{H}^0({}'A^\bullet) \longrightarrow \mathrm{H}^0(A^\bullet) \longrightarrow \mathrm{H}^0({}''A^\bullet) \xrightarrow{\partial^0} \mathrm{H}^1({}'A^\bullet) \longrightarrow \cdots .$$

Our next topic is the long exact sequence associated with a distinguished triangle of $\mathsf{D}(\mathscr{A})$. By the definition such a triangle is isomorphic to a triangle in $\mathsf{D}(\mathscr{A})$ of the form $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{b^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$. Furthermore, it is isomorphic to a triangle $A^\bullet \xrightarrow{\iota^\bullet} {}'C^\bullet_{f^\bullet} \xrightarrow{\pi^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$ as is shown in Section 4.3. In $\mathsf{K}(\mathscr{A})$, such a distinguished triangle is quasi-isomorphic to $A^\bullet \xrightarrow{\iota^\bullet} {}'C^\bullet_{f^\bullet} \xrightarrow{\pi^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$.

Extract the following split short exact sequence from (3.14)

$$0 \longrightarrow A^\bullet \xrightarrow{\iota^\bullet} {}'C^\bullet_{f^\bullet} \xrightarrow{\pi^\bullet} C^\bullet_{f^\bullet} \longrightarrow 0 \tag{4.2}$$

in $\mathsf{Co}(\mathscr{A})$. Then in $\mathscr{A}$ we obtain the long exact sequence

$$\begin{aligned} \cdots \longrightarrow \mathrm{H}^j(A^\bullet) \xrightarrow{\mathrm{H}^j(\iota^\bullet)} \mathrm{H}^j({}'C^\bullet_{f^\bullet}) \xrightarrow{\mathrm{H}^j(\pi^\bullet)} \mathrm{H}^j(C^\bullet_{f^\bullet}) \xrightarrow{\partial^j} \\ \xrightarrow{\partial^j} \mathrm{H}^{j+1}(A^\bullet) \xrightarrow{\mathrm{H}^{j+1}(\iota^\bullet)} \cdots \end{aligned} \tag{4.3}$$

where $\partial^j : \mathrm{H}^j(C^\bullet_{f^\bullet}) \to \mathrm{H}^{j+1}(A^\bullet)$ is the connecting morphism as defined in (8.9) in Chapter II. Then we will prove

$$\partial^j = \mathrm{H}^j(a^\bullet) \tag{4.4}$$

where $a^\bullet : C^\bullet_{f^\bullet} \to A[1]^\bullet$ is in the triangle $A^\bullet \to {}'C^\bullet_{f^\bullet} \to C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$. Recall that the definition of the connecting morphism ∂^j is $\partial^j(\overline{{}''c^j}) = \overline{{}'c^{j+1}}$ as in (8.9) in Chapter II, where ${}''c^j \in \ker \mathrm{d}^j_{C^\bullet} \subset C^j_{f^\bullet}$ and ${}'c^{j+1} \in \ker \mathrm{d}^{j+1}_{A^\bullet} \subset A^{j+1}$. Note that

$$\mathrm{d}^j_{C^\bullet}({}''c^j) = \mathrm{d}^j_{C^\bullet}\begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix} = \begin{pmatrix} -\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) \\ f^{j+1}(x^{j+1}) + \mathrm{d}^j_{B^\bullet}(y^j) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

as in (3.7). Namely,

$$\begin{cases} \mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) = 0 \\ f^{j+1}(x^{j+1}) + \mathrm{d}^j_{B^\bullet}(y^j) = 0. \end{cases} \tag{4.5}$$

On the other hand, recall that ${}'c^{j+1}$ in $\partial^j(\overline{{}''c^j}) = \overline{{}'c^{j+1}}$ satisfies

$$\iota^{j+1}({}'c^{j+1}) = \begin{pmatrix} -{}'c^{j+1} \\ 0 \\ 0 \end{pmatrix} = \mathrm{d}^j_{{}'C^\bullet}(c^j).$$

Those c^j and ${}''c^j$ are related as

$$\pi^j(c^j) = {}''c^j = \begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix}.$$

By Definition (3.12) of $\mathrm{d}^j_{{}'C^\bullet}$,

$$\mathrm{d}^j_{{}'C^\bullet}(c^j) = \mathrm{d}^j_{{}'C^\bullet}\begin{pmatrix} x^j \\ x^{j+1} \\ y^j \end{pmatrix} = \begin{pmatrix} \mathrm{d}^j_{A^\bullet}(x^j) - x^{j+1} \\ -\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) \\ f^{j+1}(x^{j+1}) + \mathrm{d}^j_{B^\bullet}(y^j) \end{pmatrix}. \tag{4.6}$$

From (4.5), the second and third rows are zero. In order to have

$$\begin{pmatrix} -{}'c^{j+1} \\ 0 \\ 0 \end{pmatrix} = \mathrm{d}^j_{{}'C^\bullet}\begin{pmatrix} x^j \\ x^{j+1} \\ y^j \end{pmatrix},$$

we must have $-{}'c^{j+1} = \mathrm{d}^j_{A^\bullet}(x^j) - x^{j+1}$, i.e., the first row of (4.6). Consequently, we get

$$\begin{aligned}\partial^j(\overline{{}''c^j}) &= \overline{{}'c^{j+1}} = \overline{x^{j+1} - \mathrm{d}^j_{A^\bullet}(x^j)} = \\ &= \overline{x^{j+1}} - \overline{\mathrm{d}^j_{A^\bullet}(x^j)} = \\ &= \overline{x^{j+1}} = \\ &= \mathrm{H}^j(a^\bullet)\overline{\begin{pmatrix} x^{j+1} \\ y^j \end{pmatrix}} = \\ &= \mathrm{H}^j(a^\bullet)(\overline{{}''c^j}).\end{aligned}$$

That is, a distinguished triangle induces the long exact sequence of cohomologies.

As we saw, a distinguished triangle is isomorphic to

$$A^\bullet \xrightarrow{\iota^\bullet} {}'C^\bullet_{f^\bullet} \xrightarrow{\pi^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet.$$

For the short exact sequence (4.2) in $\mathsf{Co}(\mathscr{A})$, we have the long exact sequence (4.3). Let $0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$ be an arbitrary short exact sequence of complexes in $\mathsf{Co}(\mathscr{A})$. We need quasi-isomorphisms $'g^\bullet$ and $h^\bullet$ in the following diagram.

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{g^\bullet} & C^\bullet & \xrightarrow{h^\bullet} & A[1]^\bullet \\ \| & & \uparrow{\scriptstyle 'a^\bullet} & & \uparrow{\scriptstyle 'g^\bullet} & & \| \\ A^\bullet & \xrightarrow{\iota^\bullet} & 'C^\bullet_{f^\bullet} & \xrightarrow{\pi^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet} & A[1]. \end{array} \tag{4.7}$$

Define $'g^\bullet : C^\bullet_{f^\bullet} = A[1]^\bullet \oplus B^\bullet \to C^\bullet$ in (4.7) by $'g^j(x^{j+1}, y^j) = g^j(y^j)$. Then define $h^\bullet : C^\bullet \to A[1]^\bullet$ to satisfy $h^\bullet \circ {'g^\bullet} = a^\bullet$ in (4.7). Note that since is $g^\bullet$ is epimorphic, for $z^j \in C^j$, there is $y^j \in B^j$ to satisfy $h^j(z^j) = h^j(g(y^j))$. By the above definition of $'g^\bullet$ we get

$$h^j(z^j) = h^j(g(y^j)) = h^j('g^j(x^{j+1}, y^j)) = a^j(x^{j+1}, y^j) = x^{j+1}.$$

From the distinguished triangle $A^\bullet \xrightarrow{\iota^\bullet} {'C^\bullet_{f^\bullet}} \xrightarrow{\pi^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$ we obtain the long exact sequence in (4.3). We still need to prove that $'g^\bullet : C^\bullet_{f^\bullet} \to C^\bullet$ is a quasi-isomorphism; then we can replace the long exact sequence (4.3) by the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^j(A^\bullet) \longrightarrow \mathrm{H}^j(B^\bullet) \longrightarrow \mathrm{H}^j(C^\bullet) \longrightarrow \mathrm{H}^{j+1}(A^\bullet) \longrightarrow \cdots$$

associated with $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$. (We have already proved that $'a^\bullet : {'C^\bullet_{f^\bullet}} \to B^\bullet$ is a quasi-isomorphism in (4.3).) As we noted in the above, $'g^\bullet : C^\bullet_{f^\bullet} \to C^\bullet$ is epimorphic. For the short exact sequence

$$0 \longrightarrow \ker {'g^\bullet} \longrightarrow C^\bullet_{f^\bullet} \xrightarrow{'g^\bullet} C^\bullet \longrightarrow 0 \tag{4.8}$$

in $\mathsf{Co}(\mathscr{A})$, we get the corresponding long exact sequence

$$\cdots \longrightarrow \mathrm{H}^j(\ker {'g^\bullet}) \longrightarrow \mathrm{H}^j(C^\bullet_{f^\bullet}) \longrightarrow \mathrm{H}^j(C^\bullet) \longrightarrow \mathrm{H}^{j+1}(\ker {'g^\bullet}) \longrightarrow \cdots .$$

We need to prove $\mathrm{H}^j(\ker {'g^\bullet}) = 0$ to conclude the quasi-isomorphism from $C^\bullet_{f^\bullet} \xrightarrow{'g^\bullet} C^\bullet$. First we have $\ker {'g^\bullet} = A[1]^\bullet \oplus \ker g^\bullet$. Since

$$0 \to A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \to 0$$

is exact, we have $\ker {'g^\bullet} = A[1]^\bullet \oplus \operatorname{im} f^\bullet$. The differential $\mathrm{d}^\bullet_{\ker {'g^\bullet}}$ of the complex $\ker {'g^\bullet}$:

$$\mathrm{d}^j_{\ker {'g^\bullet}} : A[1]^j \oplus \operatorname{im} f^j \to A[1]^{j+1} \oplus \operatorname{im} f^{j+1} \tag{4.9}$$

is defined as

$$\mathrm{d}^j_{\ker{}'g^\bullet}(x^{j+1}, f^j(x^j)) := (-\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}), f^{j+1}(x^{j+1} + \mathrm{d}^j_{A^\bullet}(x^j))). \quad (4.10)$$

Then we have

$$\mathrm{d}^{j+1}_{\ker{}'g^\bullet} \circ \mathrm{d}^j_{\ker{}'g^\bullet} = 0,$$

i.e., $\mathrm{im}\,\mathrm{d}^{j-1}_{\ker{}'g^\bullet} \subset \ker \mathrm{d}^j_{\ker{}'g^\bullet}$. An element $(x^{j+1}, f(x^j)) \in A[1]^j \oplus \mathrm{im}\, f^j$ of $\ker \mathrm{d}^j_{\ker{}'g^\bullet}$ satisfies $-\mathrm{d}^{j+1}_{A^\bullet}(x^{j+1}) = 0$ and $f^{j+1}(x^{j+1} + \mathrm{d}^j_{A^\bullet}(x^j)) = 0$. Since $f^\bullet$ is a monomorphism, we get

$$x^{j+1} + \mathrm{d}^j_{A^\bullet}(x^j) = 0.$$

For $(x^{j+1}, f^j(x^j)) \in \ker \mathrm{d}^j_{\ker{}'g^\bullet}$, let us compute $\mathrm{d}^{j-1}_{\ker{}'g^\bullet}$:

$$\mathrm{d}^{j-1}_{\ker{}'g^\bullet}(x^j, 0) = (-\mathrm{d}^j_{A^\bullet}(x^j), f^j(x^j + \mathrm{d}^{j-1}_{A^\bullet}(0))) = (x^{j+1}, f^j(x^j)).$$

Namely, $\ker \mathrm{d}^j_{\ker{}'g^\bullet} \subset \mathrm{im}\,\mathrm{d}^{j-1}_{\ker{}'g^\bullet}$ holds. Consequently, $\mathrm{H}^j(\ker{}'g^\bullet) = 0$, i.e., we get the isomorphism $\mathrm{H}^j(C^\bullet_{f^\bullet}) \to \mathrm{H}^j(C^\bullet)$. That is, $C^\bullet_{f^\bullet} \xrightarrow{'g^\bullet} C^\bullet$ is a quasi-isomorphism.

4.4.1 Properties of Distinguished Triangles

Distinguished triangles in $\mathsf{K}(\mathscr{A})$ have the following properties, and these properties characterize the totality of distinguished triangles.

(D.T.1) A triangle $A^\bullet \xrightarrow{1_{A^\bullet}} A^\bullet \to 0^\bullet \to A[1]^\bullet$ is a distinguished triangle.

Proof. This is because: by (D.T.4)

$$0^\bullet \xrightarrow{f^\bullet} A^\bullet \xrightarrow{1_{A^\bullet}} C^\bullet_{f^\bullet} = A^\bullet \to 0^\bullet = 0[1]^\bullet$$

is a distinguished triangle. This distinguished triangle is the triangle shifted left by 1, i.e.,

$$0[-1]^\bullet \xrightarrow{f^\bullet} A^\bullet \xrightarrow{1_{A^\bullet}} A^\bullet \to 0^\bullet$$

of the triangle $A^\bullet \xrightarrow{1^\bullet_A} A^\bullet \to 0^\bullet \to A[1]^\bullet$. The proof will be complete after (D.T.4) is proved.

(D.T.2) A triangle which is isomorphic to a distinguished triangle is also a distinguished triangle.

(D.T.3) For an arbitrary morphism $f^\bullet : A^\bullet \to B^\bullet$, there exist $C^\bullet$, $g^\bullet : B^\bullet \to C^\bullet$ and $h^\bullet : C^\bullet \to A[1]^\bullet$ so that

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A[1]^\bullet$$

is a distinguished triangle.

Proof. We have already constructed such an object $C^\bullet$ as $C^\bullet_{f^\bullet}$, morphisms $g^\bullet$, $h^\bullet$ as $b^\bullet$ and $a^\bullet$, respectively, in (3.5). Namely, $C^\bullet = C^\bullet_{f^\bullet}$, $b = \left[\begin{smallmatrix} 0^\bullet \\ 1^\bullet_{B^\bullet} \end{smallmatrix}\right]$ and $a^\bullet = [1_{A^\bullet}[1]^\bullet, 0^\bullet]$.

(D.T.4) A triangle $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A[1]^\bullet$ is a distinguished triangle if and only if

$$B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A[1]^\bullet \xrightarrow{-f[1]^\bullet} B[1]^\bullet$$

is a distinguished triangle.

Proof.

($\Rightarrow$) From (D.T.2) we may prove the statement for a distinguished triangle of the form $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{b^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet$. Then for the triangle $B^\bullet \xrightarrow{b^\bullet} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet} A[1]^\bullet \xrightarrow{-f[1]^\bullet} B[1]^\bullet$ to be distinguished it is enough to prove an isomorphism between the following triangles:

$$\begin{array}{ccccccc} B^\bullet & \xrightarrow{b^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet} & A[1]^\bullet & \xrightarrow{-f[1]^\bullet} & B[1]^\bullet \\ \| & & \| & & \downarrow{\scriptstyle \gamma^\bullet} & & \| \\ B^\bullet & \xrightarrow{b^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{c^\bullet} & C^\bullet_{b^\bullet} & \xrightarrow{'b^\bullet} & B[1]^\bullet, \end{array} \tag{4.11}$$

where $C^\bullet_{b^\bullet}$ is the cone of $b^\bullet : B^\bullet \to C^\bullet_{f^\bullet}$. Namely, $C^\bullet_{b^\bullet} := B[1]^\bullet \oplus C^\bullet_{f^\bullet}$ and the differential $\mathrm{d}^\bullet_{b^\bullet}$ on $C^\bullet_{b^\bullet}$ is given by

$$\mathrm{d}^\bullet_{b^\bullet} := \begin{bmatrix} \mathrm{d}^\bullet_{B[1]^\bullet} & 0^\bullet \\ b[1]^\bullet & \mathrm{d}^\bullet_{C^\bullet} \end{bmatrix}$$

as in (3.7). Furthermore, for $C^\bullet_{f^\bullet} = A[1]^\bullet \oplus B^\bullet$, we have

$$b[1]^\bullet = \begin{bmatrix} 0^\bullet \\ 1_{B^\bullet}[1]^\bullet \end{bmatrix} \qquad \text{and} \qquad \mathrm{d}^\bullet_{C^\bullet} = \begin{bmatrix} \mathrm{d}_{A^\bullet}[1]^\bullet & 0^\bullet \\ f[1]^\bullet & \mathrm{d}^\bullet_{B^\bullet} \end{bmatrix}.$$

More explicitly, we can write the differential $\mathrm{d}^\bullet_{b^\bullet}$ defined above, as

$$\mathrm{d}^\bullet_{b^\bullet} = \begin{bmatrix} \mathrm{d}^\bullet_{B[1]^\bullet} & 0^\bullet & 0^\bullet \\ 0^\bullet & \mathrm{d}^\bullet_{A[1]^\bullet} & 0^\bullet \\ 1_{B^\bullet}[1]^\bullet & f[1]^\bullet & \mathrm{d}^\bullet_{B^\bullet} \end{bmatrix}. \tag{4.12}$$

As we computed in (3.8) we can confirm $\mathrm{d}^{j+1}_{b^\bullet} \circ \mathrm{d}^j_{b^\bullet} = 0$.

Next we will define $\gamma^\bullet : A[1]^\bullet \to C^\bullet_{b^\bullet}$ so that $\gamma^\bullet$ becomes an isomorphism in $\mathsf{K}(\mathscr{A})$. For

$$\gamma^\bullet : A[1]^\bullet \to C^\bullet_{b^\bullet} = B[1]^\bullet \oplus C^\bullet_{f^\bullet} = B[1]^\bullet \oplus A[1]^\bullet \oplus B^\bullet,$$

if we define

$$\gamma^\bullet := \begin{bmatrix} f[1]^\bullet \\ 1_{A^\bullet}[1]^\bullet \\ 0^\bullet \end{bmatrix}, \tag{4.13}$$

$\gamma^\bullet$ is a morphism of complexes. Then $\gamma^\bullet$ also becomes a morphism of triangles, i.e., $\gamma^\bullet \circ a^\bullet = c^\bullet$ in $\mathsf{K}(\mathscr{A})$. This is because $c^\bullet$ and $\gamma^\bullet \circ a^\bullet$ are homotopic in $\mathsf{Co}(\mathscr{A})$. Namely, for $c^\bullet$ and $\gamma^\bullet \circ a^\bullet$ from $C^\bullet_{f^\bullet}$ to $C^\bullet_{b^\bullet}$:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & C^{j-1}_{f^\bullet} & \longrightarrow & C^{j}_{f^\bullet} & \longrightarrow & C^{j+1}_{f^\bullet} & \longrightarrow & \cdots \\ & & \downarrow\downarrow & \overset{s^j}{\swarrow} & \downarrow\downarrow & \overset{s^{j+1}}{\swarrow} & \downarrow\downarrow & & \\ \cdots & \longrightarrow & C^{j-1}_{b^\bullet} & \longrightarrow & C^{j}_{b^\bullet} & \longrightarrow & C^{j+1}_{b^\bullet} & \longrightarrow & \cdots , \end{array} \tag{4.14}$$

we need $s^\bullet : C^\bullet_{f^\bullet} \to C_{b^\bullet}[-1]^\bullet$ to satisfy

$$c^\bullet - \gamma^\bullet \circ a^\bullet = s[1]^\bullet \circ \mathrm{d}^\bullet_{C^\bullet} - \mathrm{d}_{b^\bullet}[-1]^\bullet \circ s^\bullet. \tag{4.15}$$

Such an $s^\bullet$ is:

$$s^\bullet := \begin{bmatrix} 0^\bullet & 1^\bullet_{B^\bullet} \\ 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet \end{bmatrix}, \tag{4.16}$$

i.e., $s^j \begin{pmatrix} a^{j+1} \\ b^j \end{pmatrix} = \begin{pmatrix} b^j \\ 0 \\ 0 \end{pmatrix}$. Then for $\begin{pmatrix} a^{j+1} \\ b^j \end{pmatrix} \in C^j_{f^\bullet}$, the right and the left hand-sides of (4.15) become

$$\begin{pmatrix} -f^{j+1}(a^{j+1}) \\ 0 \\ b^j \end{pmatrix}.$$

Consequently, $c^\bullet$ and $\gamma^\bullet \circ a^\bullet$ are homotopic, i.e., $\gamma^\bullet$ becomes a morphism of triangles. Lastly, we will prove that $\gamma^\bullet$ is an isomorphism in $\mathsf{K}(\mathscr{A})$. For $C^\bullet_{b^\bullet} = B[1]^\bullet \oplus A[1]^\bullet \oplus B^\bullet$ define $\delta^\bullet : C^\bullet_{b^\bullet} \to A[1]^\bullet$ as the projection, i.e.,

$$\delta^j \begin{pmatrix} b^{j+1} \\ a^{j+1} \\ b^j \end{pmatrix} = a^{j+1}.$$

Then we have

$$(\delta^j \circ \gamma^j)(a^{j+1}) = \delta^j \begin{pmatrix} f^{j+1}(a^{j+1}) \\ a^{j+1} \\ 0 \end{pmatrix} = a^{j+1},$$

i.e., $\delta^\bullet \circ \gamma^\bullet = 1^\bullet_{A[1]^\bullet}$. On the other hand, since we get

$$(\delta^j \circ \gamma^j) \begin{pmatrix} b^{j+1} \\ a^{j+1} \\ b^j \end{pmatrix} = \gamma^{j+1}(a^{j+1}) = \begin{pmatrix} f^{j+1}(a^{j+1}) \\ a^{j+1} \\ 0 \end{pmatrix},$$

we look for a homotopy morphism $t^j : C^j_{b^\bullet} \to C^{j-1}_{b^\bullet}$ satisfying

$$\gamma^\bullet \circ \delta^\bullet - 1^\bullet_{C^\bullet_{b^\bullet}} = t[1]^\bullet \circ \mathrm{d}^\bullet_{b^\bullet} - \mathrm{d}_{b^\bullet}[-1]^\bullet \circ t^\bullet.$$

Choose such a $t^\bullet$ as

$$t^\bullet := \begin{bmatrix} 0^\bullet & 0^\bullet & 1^\bullet_{B^\bullet} \\ 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet \end{bmatrix}.$$

Then we can confirm that $\gamma^\bullet \circ \delta^\bullet$ is indeed homotopic to $1^\bullet_{C^\bullet_{b^\bullet}}$.

($\Leftarrow$) The converse of (D.T.4) can be proved by the repeated use (e.g., six times) of the above first half of the assertion. That is, we get the distinguished triangle

$$A[2]^\bullet \xrightarrow{f[2]^\bullet} B[2]^\bullet \xrightarrow{g[2]^\bullet} C[2]^\bullet \xrightarrow{h[2]^\bullet} A[3]^\bullet$$

which is isomorphic to $A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A[1]^\bullet$.

Recall that the proof of (D.T.1) can be completed by (D.T.4).

(D.T.5) For two distinguished triangles

$$A^\bullet \xrightarrow{f^\bullet} B^\bullet \xrightarrow{g^\bullet} C^\bullet \xrightarrow{h^\bullet} A[1]^\bullet$$

and

$$'A^\bullet \xrightarrow{'f^\bullet} {'B^\bullet} \xrightarrow{'g^\bullet} {'C^\bullet} \xrightarrow{'h^\bullet} {'A[1]^\bullet},$$

if $\alpha^\bullet : A^\bullet \to {'A^\bullet}$ and $\beta^\bullet : B^\bullet \to {'B^\bullet}$ are given satisfying $'f^\bullet \circ \alpha^\bullet = \beta^\bullet \circ f^\bullet$, then there exists $\gamma^\bullet : C^\bullet \to {'C^\bullet}$ making the diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f^\bullet} & B^\bullet & \xrightarrow{g^\bullet} & C^\bullet & \xrightarrow{h^\bullet} & A[1]^\bullet \\ \downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle \beta^\bullet} & & \downarrow{\scriptstyle \gamma^\bullet} & & \downarrow{\scriptstyle \alpha[1]^\bullet} \\ 'A^\bullet & \xrightarrow{'f^\bullet} & 'B^\bullet & \xrightarrow{'g^\bullet} & 'C^\bullet & \xrightarrow{'h^\bullet} & 'A[1]^\bullet \end{array} \tag{4.17}$$

commutative.

Proof. It is enough to prove the commutativity of diagram (4.17) for the case where $C^\bullet = C^\bullet_{f^\bullet}$ and $'C^\bullet = C^\bullet_{'f^\bullet}$. Then we can let

$$\gamma^\bullet := \alpha[1]^\bullet \oplus \beta^\bullet : C^\bullet_{f^\bullet} \to C^\bullet_{'f^\bullet}$$

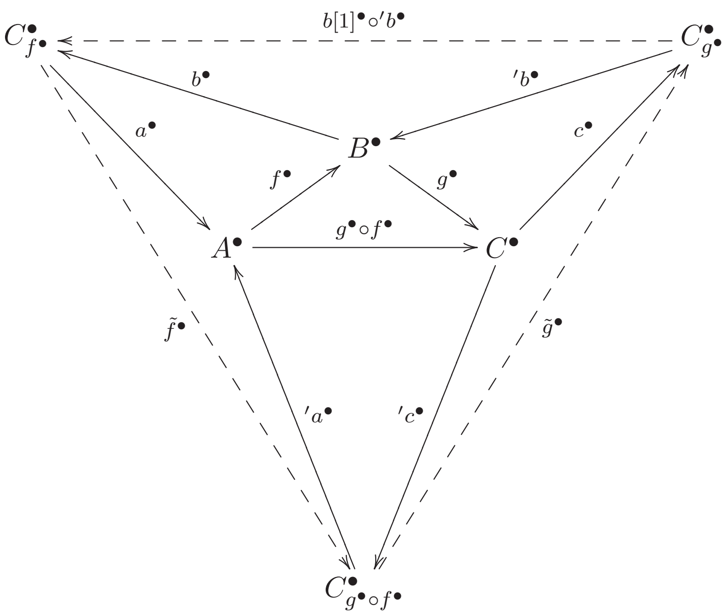
to get $b^\bullet \circ \beta^\bullet = (\alpha[1]^\bullet \oplus \beta^\bullet) \circ {}'b^\bullet$ where $b^\bullet : B^\bullet \to C^\bullet_{f^\bullet}$ and $'b^\bullet : {}'B^\bullet \to C^\bullet_{'f^\bullet}$ defined by

$$b^\bullet := \begin{bmatrix} 0^\bullet \\ 1^\bullet_{B^\bullet} \end{bmatrix}, \qquad \text{and} \qquad {}'b^\bullet := \begin{bmatrix} 0^\bullet \\ 1^\bullet_{'B^\bullet} \end{bmatrix},$$

respectively.

4.4.2 Property (D.T.6) of Distinguished Triangle

Let $f^\bullet : A^\bullet \to B^\bullet$ and $g^\bullet : B^\bullet \to C^\bullet$ be given. Then by (D.T.3) we get distinguished triangles corresponding to $f^\bullet$ and $g^\bullet$. Also for $g^\bullet \circ f^\bullet : A^\bullet \to C^\bullet$ we have a distinguished triangle. That is, for the middle commutative triangular diagram, we obtain three distinguished triangles



(4.18)

Then we assert:

(D.T.6) There exist $\tilde{f}^\bullet : C^\bullet_{f^\bullet} \to C^\bullet_{g^\bullet \circ f^\bullet}$ and $\tilde{g}^\bullet : C^\bullet_{g^\bullet \circ f^\bullet} \to C^\bullet_{g^\bullet}$ to form the distinguished triangle

$$C^\bullet_{f^\bullet} \xrightarrow{\tilde{f}^\bullet} C^\bullet_{g^\bullet \circ f^\bullet} \xrightarrow{\tilde{g}^\bullet} C^\bullet_{g^\bullet} \xrightarrow{b[1]^\bullet \circ {}'b^\bullet} C_{f^\bullet}[1]^\bullet \tag{4.19}$$

as shown in (4.18) and satisfying $a^\bullet = {}'a^\bullet \circ \tilde{f}^\bullet$ and $c^\bullet = \tilde{g}^\bullet \circ {}'c^\bullet$.

Then we also have $f^\bullet \circ {'a^\bullet} = {'b^\bullet} \circ \tilde{g}^\bullet$ and $\tilde{f}^\bullet \circ b^\bullet = {'c^\bullet} \circ g^\bullet$.

Proof. We need to find such an $\tilde{f}^\bullet : C^\bullet_{f^\bullet} \to C^\bullet_{g^\bullet \circ f^\bullet}$ so as to make $C^\bullet_{g^\bullet}$ and $C^\bullet_{\tilde{f}^\bullet}$ isomorphic. Since $C^\bullet_{f^\bullet} = A[1]^\bullet \oplus B^\bullet$ and $C^\bullet_{g^\bullet \circ f^\bullet} = A[1]^\bullet \oplus C^\bullet$, the natural choice for $\tilde{f}^\bullet$ is

$$\tilde{f}^\bullet := \begin{bmatrix} 1_{A^\bullet}[1]^\bullet & 0^\bullet \\ 0^\bullet & g^\bullet \end{bmatrix}, \tag{4.20}$$

i.e., $\tilde{f}^j(a^{j+1}, b^j) = (a^{j+1}, g^j(b^j))$. Since $\tilde{g}^\bullet$ is from $C^\bullet_{g^\bullet \circ f^\bullet} = A[1]^\bullet \oplus C^\bullet$ to $C^\bullet_{g^\bullet} = B[1]^\bullet \oplus C^\bullet$, $\tilde{g}^\bullet$ should be

$$\tilde{g}^\bullet := \begin{bmatrix} f[1]^\bullet & 0^\bullet \\ 0^\bullet & 1^\bullet_{C^\bullet} \end{bmatrix}. \tag{4.21}$$

Let us prove that there is an isomorphism between the following triangles:

$$\begin{array}{ccccccc}
C^\bullet_{f^\bullet} & \xrightarrow{\tilde{f}^\bullet} & C^\bullet_{g^\bullet \circ f^\bullet} & \xrightarrow{\tilde{g}^\bullet} & C^\bullet_{g^\bullet} & \xrightarrow{b[1]^\bullet \circ {'b^\bullet}} & C_{f^\bullet}[1]^\bullet \\
\| & & \| & & \downarrow{\scriptstyle \delta^\bullet} & & \| \\
C^\bullet_{f^\bullet} & \xrightarrow{\tilde{f}^\bullet} & C^\bullet_{g^\bullet \circ f^\bullet} & \xrightarrow{{}^*c^\bullet} & C^\bullet_{\tilde{f}^\bullet} & \xrightarrow{''c^\bullet} & C_{f^\bullet}[1]^\bullet
\end{array}$$

Since $\delta^\bullet$ is from $C^\bullet_{g^\bullet} = B[1]^\bullet \oplus C$ to

$$\begin{aligned} C^\bullet_{\tilde{f}^\bullet} = C_{f^\bullet}[1]^\bullet \oplus C^\bullet_{g^\bullet \circ f^\bullet} &= (A[1]^\bullet \oplus B^\bullet)[1] \oplus (A[1]^\bullet \oplus C^\bullet) = \\ &= A[2]^\bullet \oplus B[1]^\bullet \oplus A[1]^\bullet \oplus C^\bullet, \end{aligned}$$

$\delta^\bullet$ needs to be an identity morphism. To be precise,

$$\delta^\bullet := \begin{bmatrix} 0^\bullet & 0^\bullet \\ 1_{B^\bullet}[1]^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet \\ 0^\bullet & 1^\bullet_{C^\bullet} \end{bmatrix}.$$

On the other hand

$$'\delta^\bullet : C^\bullet_{\tilde{f}^\bullet} = A[2]^\bullet \oplus B[1]^\bullet \oplus A[1]^\bullet \oplus C^\bullet \longrightarrow C^\bullet_{g^\bullet} = B[1]^\bullet \oplus C^\bullet$$

needs to be defined so that the image $(0, b^{j+1}, 0, c^j)$ of (b^{j+1}, c^j) by $\delta^\bullet$ may be (b^{j+1}, c^j) by $'\delta^\bullet$. Hence, let

$$'\delta^\bullet := \begin{bmatrix} 0^\bullet & 1_{B^\bullet}[1]^\bullet & f[1]^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 1^\bullet_{C^\bullet} \end{bmatrix}.$$

Then we have ${}'\delta^\bullet \circ \delta^\bullet = 1^\bullet_{C^\bullet_{g^\bullet}}$. Since $\delta^\bullet \circ {}'\delta^\bullet$ does not equal $1^\bullet_{C^\bullet_{\tilde{f}^\bullet}}$ we need to prove that $\delta^\bullet \circ {}'\delta^\bullet$ is homotopic to $1^\bullet_{C^\bullet_{\tilde{f}^\bullet}}$. If we define $s^j : C^j_{\tilde{f}^\bullet} \to C^{j-1}_{\tilde{f}^\bullet}$ by

$$s^\bullet := \begin{bmatrix} 0^\bullet & 0^\bullet & 1_{A^\bullet}[1]^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 0^\bullet \\ 0^\bullet & 0^\bullet & 0^\bullet & 0^\bullet \end{bmatrix},$$

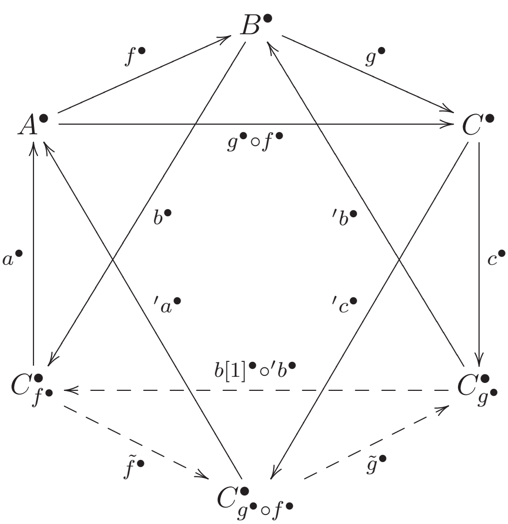
then we obtain

$$1^\bullet_{C^\bullet_{\tilde{f}^\bullet}} - \delta^\bullet \circ {}'\delta^\bullet = s[1]^\bullet \circ \mathrm{d}^\bullet_{\tilde{f}^\bullet} - \mathrm{d}_{\tilde{f}^\bullet}[-1]^\bullet \circ s^\bullet.$$

That is, in $\mathsf{K}(\mathscr{A})$, $\delta^\bullet$ is an isomorphism. One can confirm that ${}'\delta^\bullet$ is a morphism of triangles, i.e., $\tilde{g}^\bullet = {}'\delta^\bullet \circ {}^*c^\bullet$. Confirm also that the commutativity of all the triangular diagrams in (4.18). Finally, two-way two paths connecting $B^\bullet$ and $C^\bullet_{g^\bullet \circ f^\bullet}$ also satisfy the commutativity.

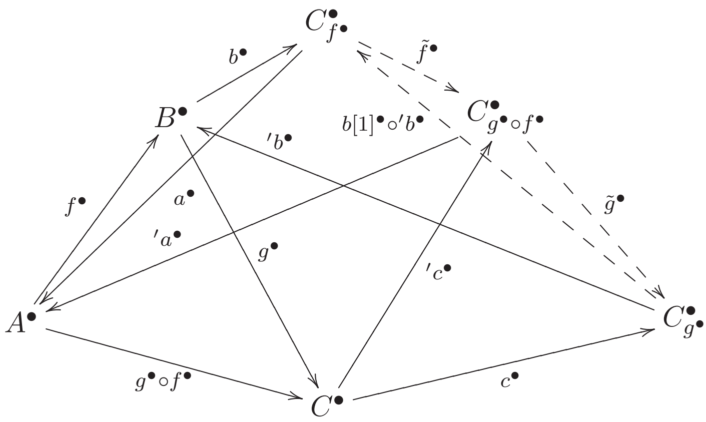
4.4.3 Remarks on Diagram (4.18)

There are other ways to write diagram (4.18) for (D.T.6). For example, we can write (4.18) as



(4.22)

or as



(4.23)

The property (or axiom) (D.T.6) is said to be the *octahedral property* (or *axiom*) because of the octahedral shape of the diagram (4.22).

4.4.4 Distinguished Triangles in Derived Categories

First we will define a distinguished triangle in the derived category $\mathsf{D}(\mathscr{A})$ via the notion of a distinguished triangle in $\mathsf{K}(\mathscr{A})$. Let $\phi^\bullet : A^\bullet \to B^\bullet$ and $\psi^\bullet : B^\bullet \to C^\bullet$ be morphisms in $\mathsf{D}(\mathscr{A})$. For a morphism $\lambda^\bullet : C^\bullet \to A[1]^\bullet$ we have the triangle

$$A^\bullet \xrightarrow{\phi^\bullet} B^\bullet \xrightarrow{\psi^\bullet} C^\bullet \xrightarrow{\lambda^\bullet} A[1]^\bullet \tag{4.24}$$

in $\mathsf{D}(\mathscr{A})$. Then the triangle (4.24) is said to be a *distinguished triangle* in $\mathsf{D}(\mathscr{A})$ when the following are satisfied: for a distinguished triangle

$${}'A^\bullet \xrightarrow{f^\bullet} {}'B^\bullet \xrightarrow{g^\bullet} {}'C^\bullet \xrightarrow{h^\bullet} {}'A[1]^\bullet \tag{4.25a}$$

in $\mathsf{K}(\mathscr{A})$ there is an isomorphism of $\mathsf{D}(\mathscr{A})$ from triangle (4.25a) to triangle (4.24). That is, for the localizing functor

$$\mathsf{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$$

as in (1.7), the triangle

$${}'A^\bullet \xrightarrow{\mathsf{Q}_{\mathscr{A}} f^\bullet} {}'B^\bullet \xrightarrow{\mathsf{Q}_{\mathscr{A}} g^\bullet} {}'C^\bullet \xrightarrow{\mathsf{Q}_{\mathscr{A}} h^\bullet} {}'A[1]^\bullet \tag{4.25b}$$

in $\mathsf{D}(\mathscr{A})$ is isomorphic to triangle (4.24).

Let us verify some of the properties (D.T.1) through (D.T.6). Let

$$\phi^\bullet : A^\bullet \to B^\bullet$$

be a morphism in $\mathsf{D}(\mathscr{A})$. Choose a representative $(f^\bullet, s^\bullet)$ of $\phi^\bullet = f^\bullet/s^\bullet$ as in (1.12). Namely, we have

$$\begin{array}{ccc} & & {}'B^\bullet \\ & \overset{f^\bullet}{\nearrow} & \uparrow{\scriptstyle s^\bullet} \\ A^\bullet & \xrightarrow[\phi^\bullet]{} & B^\bullet, \end{array}$$

which give us the following diagram

$$\begin{array}{ccccccc} A^\bullet & \xrightarrow{f^\bullet} & {}'B^\bullet & \xrightarrow{{}'b^\bullet} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet} & A[1]^\bullet \\ \uparrow{\scriptstyle 1^\bullet_{A^\bullet}} & & \uparrow{\scriptstyle s^\bullet} & & \uparrow{\scriptstyle 1^\bullet_{C^\bullet_{f^\bullet}}} & & \uparrow{\scriptstyle 1^\bullet_{A[1]^\bullet}} \\ A^\bullet & \xrightarrow{\phi^\bullet} & B^\bullet & \xrightarrow{{}'b^\bullet\circ s^\bullet/1_{C^\bullet_{f^\bullet}}} & C^\bullet_{f^\bullet} & \xrightarrow{a^\bullet/1^\bullet_{A[1]^\bullet}} & A[1]^\bullet. \end{array} \tag{4.26a}$$

The first row of (4.26a) becomes a distinguished triangle in $\mathsf{K}(\mathscr{A})$ by the construction of the mapping cone of $f^\bullet$ in (3.5). Then the functor $\mathsf{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ takes this distinguished triangle to the triangle in $\mathsf{D}(\mathscr{A})$

$$A^\bullet \xrightarrow{f^\bullet/1_{'B^\bullet}} {}'B^\bullet \xrightarrow{{}'b^\bullet/1^\bullet_{C^\bullet_{f^\bullet}}} C^\bullet_{f^\bullet} \xrightarrow{a^\bullet/1_{A[1]^\bullet}} A[1]^\bullet \tag{4.26b}$$

as $\mathsf{Q}_{\mathscr{A}} s^\bullet$ becomes an isomorphism in $\mathsf{D}(\mathscr{A})$. Namely, the triangle of $\mathsf{D}(\mathscr{A})$ in the second row of (4.26a) and the above triangle (4.26b) are isomorphic. That is, an arbitrary morphism $\phi^\bullet : A^\bullet \to B^\bullet$ of the derived category $\mathsf{D}(\mathscr{A})$ can be embedded into a distinguished triangle, i.e., property (D.T.3) of Subsection 4.4.1.

Next, let us verify (D.T.6) for the derived category $\mathsf{D}(\mathscr{A})$. In $\mathsf{D}(\mathscr{A})$, let $\phi^\bullet : A^\bullet \to B^\bullet$ and $\psi^\bullet : B^\bullet \to C^\bullet$ be morphisms. Then we have

$$\psi^\bullet \circ \phi^\bullet : A^\bullet \to C^\bullet.$$

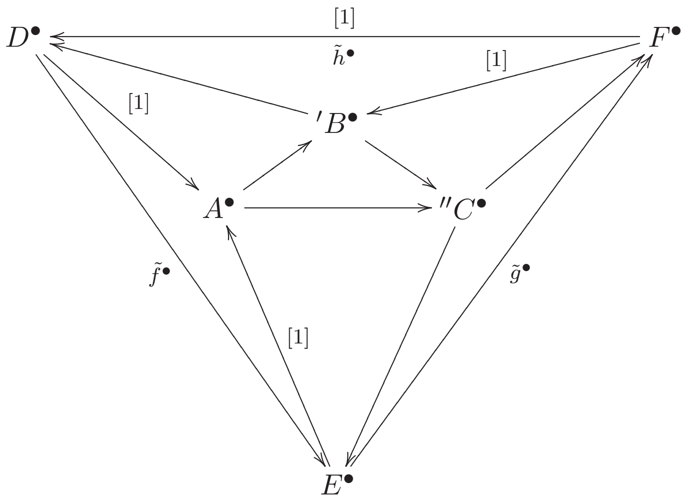
We also let $\phi^\bullet = f^\bullet/s^\bullet$ and $\psi^\bullet = g^\bullet/t^\bullet$ as in (1.13). Then from (1.13), we get

$$\begin{array}{ccccc} & & B^\bullet & & \\ & \overset{f^\bullet/s^\bullet}{\nearrow} & & \overset{g^\bullet/t^\bullet}{\searrow} & \\ A^\bullet & & \xrightarrow[{}'g^\bullet\circ f^\bullet/{}'s^\bullet\circ t^\bullet]{} & & C^\bullet. \end{array} \tag{4.27}$$

By using the notation in (1.13), we have in $\mathsf{K}(\mathscr{A})$

$$\begin{array}{ccccc} & & {}'B^\bullet & & \\ & \overset{f^\bullet}{\nearrow} & & \overset{{}'g^\bullet}{\searrow} & \\ A^\bullet & & \xrightarrow[{}'g^\bullet\circ f^\bullet]{} & & {}''C^\bullet. \end{array} \tag{4.28}$$

Note that there are quasi-isomorphisms $s^\bullet : B^\bullet \to {}'B^\bullet$ and ${}'s^\bullet \circ t^\bullet : C^\bullet \to {}''C^\bullet$. By (D.T.6) for $\mathsf{K}(\mathscr{A})$ in Subsection 4.4.2, for the three distinguished triangles corresponding to $f^\bullet, {}'g^\bullet$ and ${}'g^\bullet \circ f^\bullet$

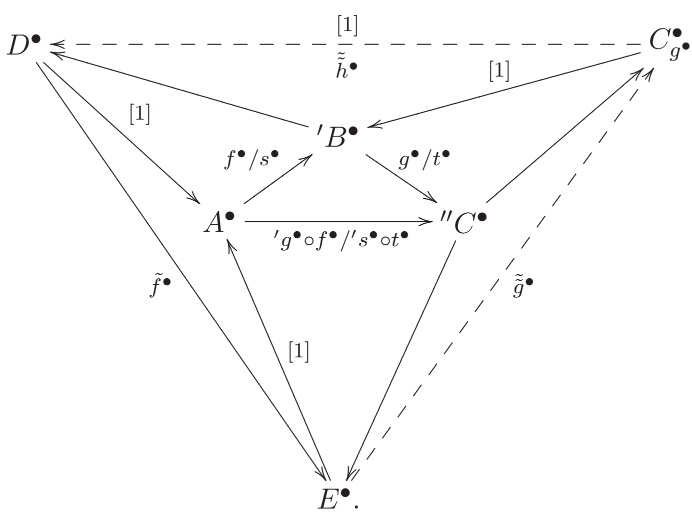


(4.29)

we have the distinguished triangle in $\mathsf{K}(\mathscr{A})$

$$D^\bullet \xrightarrow{\tilde{f}^\bullet} E^\bullet \xrightarrow{\tilde{g}^\bullet} F^\bullet \xrightarrow{\tilde{h}^\bullet} D[1]^\bullet, \tag{4.30}$$

as in (4.29), where $D^\bullet = C^\bullet_{f^\bullet}$, $E^\bullet = C^\bullet_{'g^\bullet \circ f^\bullet}$ and $F^\bullet = C^\bullet_{'g^\bullet}$. By the construction of the distinguished triangle in $\mathsf{D}(\mathscr{A})$ for a morphism, i.e., (4.27), we have in $\mathsf{D}(\mathscr{A})$



(4.31)

Since $C^\bullet_{g^\bullet} = B[1]^\bullet \oplus C^\bullet$ and $C^\bullet_{'g^\bullet} = {}'B[1]^\bullet \oplus {}''C^\bullet$ and since $B^\bullet \to {}'B^\bullet$ and $C^\bullet \to {}''C^\bullet$ are quasi-isomorphisms, the functor $\mathsf{Q}_{\mathscr{A}} : \mathsf{K}(\mathscr{A}) \rightsquigarrow \mathsf{D}(\mathscr{A})$ takes the distinguished triangle (4.29) to the triangle isomorphic to the triangle in $\mathsf{D}(\mathscr{A})$

$$D^\bullet = C^\bullet_{f^\bullet} \xrightarrow{\tilde{f}^\bullet} E^\bullet = C^\bullet_{'g^\bullet \circ f^\bullet} \xrightarrow{\tilde{\tilde{g}}^\bullet} C^\bullet_{g^\bullet} \approx C^\bullet_{'g^\bullet} \xrightarrow{\tilde{\tilde{h}}^\bullet} D[1]^\bullet, \tag{4.32}$$

where, explicitly,

$$\tilde{\tilde{g}}^\bullet = (s[1]^\bullet \oplus ({}'s^\bullet \circ t^\bullet))^{-1} \circ \tilde{g}^\bullet$$

and

$$\tilde{\tilde{h}}^\bullet = ({}'b^\bullet \circ s^\bullet / 1^\bullet_{C_{f^\bullet}}) \circ (b^\bullet / 1^\bullet_{B[1]^\bullet})$$

as in (4.26a). Namely, the triangle in (4.31)

$$D^\bullet \xrightarrow{\tilde{f}^\bullet} E^\bullet \xrightarrow{\tilde{\tilde{g}}^\bullet} C^\bullet_{g^\bullet} \xrightarrow{\tilde{\tilde{h}}^\bullet} D[1]^\bullet$$

is distinguished in $\mathsf{D}(\mathscr{A})$, i.e., (D.T.6) for $\mathsf{D}(\mathscr{A})$.

As for (D.T.4), the corresponding claim of (D.T.4) for $\mathsf{D}(\mathscr{A})$ follows from the diagram in (4.26a). Namely,

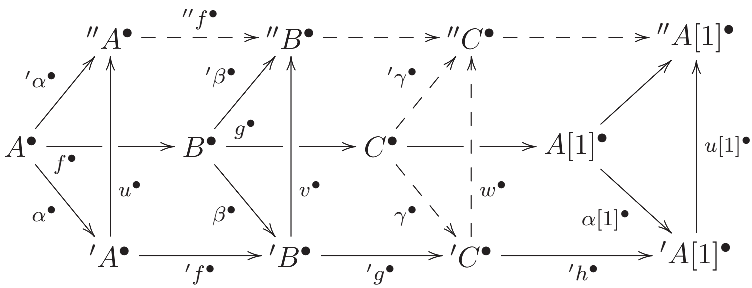
$$\begin{array}{ccccccc}
{}'B^\bullet & \longrightarrow & C^\bullet_{f^\bullet} & \longrightarrow & A[1]^\bullet & \xrightarrow{-f[1]^\bullet} & {}'B[1]^\bullet \\
{\scriptstyle s^\bullet}\uparrow & & {\scriptstyle 1^\bullet}\uparrow & & {\scriptstyle 1^\bullet}\uparrow & & {\scriptstyle s[1]^\bullet}\uparrow \\
B^\bullet & \longrightarrow & C^\bullet_{f^\bullet} & \longrightarrow & A[1]^\bullet & \xrightarrow{-\phi[1]^\bullet} & B[1]^\bullet
\end{array} \tag{4.33}$$

if the first row triangle is distinguished in $\mathsf{K}(\mathscr{A})$ then since $s^\bullet$ is a quasi-isomorphism, the second row triangle of (4.33) is a distinguished triangle in $\mathsf{D}(\mathscr{A})$. The converse is also confirmed in a similar way.

We will now confirm (D.T.5). That is, for two distinguished triangles in $\mathsf{D}(\mathscr{A})$ and morphisms $\alpha^\bullet$ and $\beta^\bullet$ of $\mathsf{D}(\mathscr{A})$

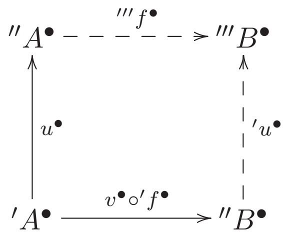
$$\begin{array}{ccccccc}
A^\bullet & \xrightarrow{\phi^\bullet} & B^\bullet & \xrightarrow{\psi^\bullet} & C^\bullet & \xrightarrow{\lambda^\bullet} & A[1]^\bullet \\
\downarrow{\scriptstyle \alpha^\bullet} & & \downarrow{\scriptstyle \beta^\bullet} & & \vdots{\scriptstyle \gamma^\bullet} & & \downarrow{\scriptstyle \alpha[1]^\bullet} \\
{}'A^\bullet & \xrightarrow{'\phi^\bullet} & {}'B^\bullet & \xrightarrow{'\psi^\bullet} & {}'C^\bullet & \xrightarrow{'\lambda^\bullet} & {}'A[1]^\bullet
\end{array} \tag{4.34}$$

satisfying ${}'\phi^\bullet \circ \alpha^\bullet = \beta^\bullet \circ \phi^\bullet$, we need to construct $\gamma^\bullet : C^\bullet \to {}'C^\bullet$ to satisfy ${}'\psi^\bullet \circ \beta^\bullet = \gamma^\bullet \circ \psi^\bullet$. First, replace the above distinguished triangles of $\mathsf{D}(\mathscr{A})$ by distinguished triangles of $\mathsf{K}(\mathscr{A})$ that are mutually isomorphic in $\mathsf{D}(\mathscr{A})$ as a pair. Let $\alpha^\bullet := {}'\alpha^\bullet/u^\bullet$ and $\beta^\bullet := {}'\beta^\bullet/v^\bullet$. Consider the following diagram:



(4.35)

where $u^\bullet$ and $v^\bullet$ are quasi-isomorphisms. Then by (QIS.2) in Subsection 4.1.2, for



(4.36)

the far right square extracted from the above (4.35), we can have a quasi-isomorphism ${}'u^\bullet$ and a morphism ${}'''f^\bullet$ as indicated in (4.36). Since ${}'u^\bullet$ and $v^\bullet$ are both quasi-isomorphisms, we may claim that there exists an

$$''f^\bullet : {}''A^\bullet \to {}''B^\bullet$$

in (4.35) making the far right square commutative. By (D.T.3) for $\mathsf{K}(\mathscr{A})$ we get a distinguished triangle

$$''A^\bullet \xrightarrow{\;''f^\bullet\;} {}''B^\bullet \xrightarrow{\;''g^\bullet\;} {}''C^\bullet \xrightarrow{\;''h^\bullet\;} {}''A[1]^\bullet.$$

Then by (D.T.5) for $\mathsf{K}(\mathscr{A})$ we obtain ${}'\gamma^\bullet : C^\bullet \to {}''C^\bullet$ as in (4.35) and also by (D.T.5) for $\mathsf{K}(\mathscr{A})$, we have a quasi-isomorphism $w^\bullet : C^\bullet \to {}''C^\bullet$ as in (4.35). Then let $\gamma^\bullet := {}'\gamma^\bullet/w^\bullet : C^\bullet \to {}'C^\bullet$ as in (4.35). (Note that the details of the proof are left to be completed.)

Chapter 5

COHOMOLOGICAL ASPECTS OF ALGEBRAIC GEOMETRY AND ALGEBRAIC ANALYSIS

5.1 Exposition

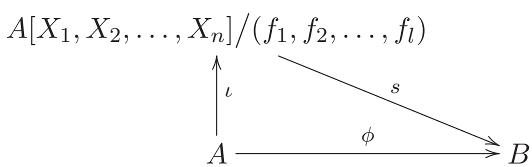
The most fundamental object of study in Algebraic Geometry is the set of (or the number) of solutions of a system of (polynomial) equations. That is, for a system of equations

$$\begin{cases} f_1(x_1, x_2, \ldots, x_n) = 0 \\ f_2(x_1, x_2, \ldots, x_n) = 0 \\ \quad\vdots \\ f_l(x_1, x_2, \ldots, x_n) = 0 \end{cases} \tag{1.1}$$

with coefficients in a commutative ring A with identity, we seek for solutions in an algebra B over A, i.e., $(b_1, b_2 \ldots, b_n) \in B^n := \underbrace{B \times B \times \cdots \times B}_{n}$. Such a solution is said to be a B*-rational point*. In terms of commutative algebra we can rephrase the above as follows. For a finitely generated A-algebra C,

$$A[X_1, X_2, \ldots, X_n] \xrightarrow{\Psi} C \to 0,$$

where the A-algebra homomorphism is the canonical one defined by $\Psi(X_k) = x_k$, $1 \le k \le n$, a B-rational point is an A-algebra homomorphism s to make the diagram



(1.2)

commutative, where the A-algebra homomorphism ϕ is defining the algebra structure of B. Namely, for the finitely generated A-algebra C

$$\begin{array}{ccc} C = A[x_1, x_2, \ldots, x_n] & \xleftarrow{\approx} & A[X_1, X_2, \ldots, X_n]/\ker\Psi \\ & & \Vert \\ & & A[X_1, X_2, \ldots, X_n]/(f_1, f_2, \ldots, f_l) \end{array}$$

the set $\mathrm{Hom}_A(C, B)$ of all the A-algebra homomorphisms is the set of all B-rational points of C. That is, for $s \in \mathrm{Hom}_A(C, B)$, the commutative diagram gives

$$\begin{aligned} s(f_k(x_1, x_2, \ldots, x_n)) &= s(\sum a_{i_1 \cdots i_n} x_1^{i_1} x_2^{i_2} \cdots x_n^{i_n}) = \\ &= \sum a_{i_1 \cdots i_n} s(x_1)^{i_1} s(x_2)^{i_2} \cdots s(x_n)^{i_n} = \\ &= \sum a_{i_1 \cdots i_n} b_1^{i_1} b_2^{i_2} \cdots b_n^{i_n} = 0. \end{aligned} \tag{1.3}$$

In general for a scheme X over a commutative ring A, the set of scheme morphisms from an A-algebra B to X is the set of B-rational points on X:

$$\begin{array}{ccc} X & \xleftarrow{\quad s \quad} & \\ \downarrow & & \\ \mathrm{Spec}\, A & \longleftarrow & \mathrm{Spec}\, B. \end{array} \tag{1.4}$$

We often write $X(B)$ for the set of B-rational points on X. Compare the above rational point notation with the formulation of Yoneda's Lemma in Chapter I.

For example, let $\mathbb{Z}$ be the ring of integers and let p be a prime. From

$$\mathbb{Z} \supset p\mathbb{Z} \supset p^2\mathbb{Z} \supset \cdots, \tag{1.5}$$

we get the sequence

$$\cdots \longrightarrow \mathbb{Z}/p^3\mathbb{Z} \longrightarrow \mathbb{Z}/p^2\mathbb{Z} \longrightarrow \mathbb{Z}/p\mathbb{Z}. \tag{1.6}$$

The inverse limit of (1.6) is said to be the ring of *p-adic integers* denoted as $\hat{\mathbb{Z}}_p$. We have the fan as in (8.3) in Chapter I:

$$\begin{array}{ccc} & & \mathbb{Z}/p^{n+1}\mathbb{Z} \\ & \nearrow^{\alpha_{n+1}} & \\ \hat{\mathbb{Z}}_p & & \downarrow \\ & \searrow_{\alpha_n} & \\ & & \mathbb{Z}/p^n\mathbb{Z}. \end{array} \tag{1.7}$$

For a $\hat{\mathbb{Z}}_p$-rational point s on a scheme X, i.e., $s \in X(\hat{\mathbb{Z}}_p)$, the composition $s \circ \operatorname{Spec} \alpha_n$ of the morphisms in

$$\begin{array}{ccc} X & \xleftarrow{\;s\circ\operatorname{Spec}\alpha_n\;} & \\ \uparrow{\scriptstyle s} & & \\ \operatorname{Spec}\hat{\mathbb{Z}}_p & \xleftarrow{\;\operatorname{Spec}\alpha_n\;} & \operatorname{Spec}\mathbb{Z}/p^n\mathbb{Z} \end{array} \tag{1.8}$$

gives a solution in $\mathbb{Z}/p^n\mathbb{Z}$, i.e., a $\mathbb{Z}/p^n\mathbb{Z}$-rational point on X for every $n \geq 1$. we simply write, e.g., $\hat{\mathbb{Z}}_p$ for $\operatorname{Spec} \hat{\mathbb{Z}}_p$.

More generally, let us consider the system of l-homogeneous polynomials $f_1, f_2, \ldots, f_l$ with coefficients in a finite field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ and let $I = (f_1, f_2, \ldots, f_l)$ be the ideal in $\mathbb{F}_p[X_1, X_2, \ldots, X_n]$ generated by the f_i. Then let $\bar{U}(\mathbb{F}_p)$ be the set of $\mathbb{F}_p$-rational points on $\bar{U} = \operatorname{Proj}(\mathbb{F}_p[X_1, X_2, \ldots, X_n]/I)$. Define also

$$\begin{aligned} N_k = |\bar{U}(\mathbb{F}_{p^k})| &= \text{the number of } \mathbb{F}_{p^k}\text{-rational points on } \bar{U}, \\ &\quad \text{i.e., the number of morphisms in } \operatorname{Hom}_{\mathbb{F}_p}(\mathbb{F}_{p^k}, \bar{U}) \end{aligned} \tag{1.9}$$

where $\mathbb{F}_{p^k}$ is the extension field of $\mathbb{F}_p$ of degree k. Then the zeta function $Z_{\bar{U}}$ of the projective variety $\bar{U}$ over $\mathbb{F}_p$ is defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}T}(\log Z_{\bar{U}}(T)) = \sum_{k=0}^{\infty} N_{k+1} T^k, \\ \text{where } Z_{\bar{U}}(0) = 1. \end{cases} \tag{1.10}$$

General conjectures on the zeta function associated with an algebraic variety defined over a finite field appeared in

* Weil, A., *Numbers of Solutions of Equations in Finite Fields*, Bull. Amer. Math. Soc. **55**, (1949), 297–508.

For cohomology theory and the Weil conjectures, see

* *Motives*, Part 1, Proceedings of Symposia in Pure Mathematics, **55**, the AMS, (1994).

5.2 The Weierstrass Family

We will review some of the basic facts about the Weierstrass family. The Weierstrass equation

$$Y^2 = 4X^3 - g_2 X - g_3 \tag{2.1}$$

is obtained from the cubic equations

$$Y^2 = aX^3 + bX^2 + cX + d, \qquad a \neq 0. \tag{2.2}$$

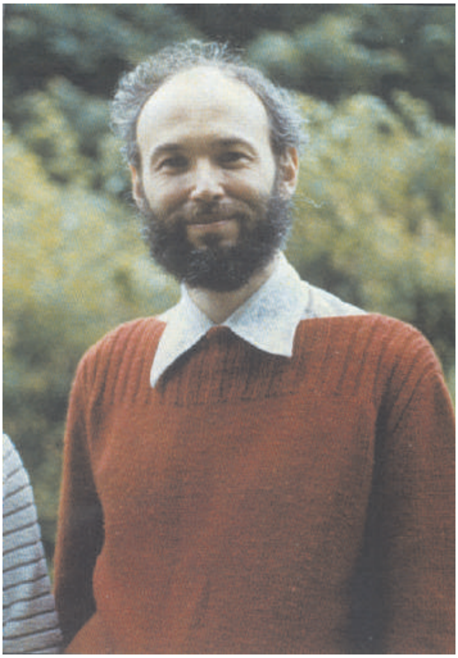


Figure 5.1. Deligne and the author's shoulder, at the IAS (the Institute for Advanced Study), 1986

Figure 5.2. Lubkin, Weil and the author's shoulder, at the IAS, Princeton, 1986

That is, let us introduce a new variable X_0 through a linear substitution $X_0 := X + \frac{b}{3a}$. We can assume $b = 0$ in (2.2). Namely, we have reduced the general cubic equation as (2.2) to

$$Y^2 = aX^3 + cX + d.$$

Making the linear substitutions $Y_0 = aY$ and $X_0 = aX$ we get

$$Y_0^2 = X_0^2 + acX_0 + a^2 d, \qquad a \neq 0.$$

Therefore, the general cubic equation reduces to the following

$$Y^2 = X^3 + aX + d. \tag{2.3}$$

Furthermore, if the characteristic is not equal to 2, the linear change $Y_0 = \frac{Y}{2}$ would give

$$Y_0^2 = X^3 + \frac{c}{4}X + \frac{d}{4}.$$

Namely, the Weierstrass equation is a normalization by linear changes of coordinates of the cubic equation $Y^2 = aX^3 + bX^2 + cX + d$.

In terms of schemes (i.e., geometric terms), we can rephrase the above argument as follows. Let R be a (commutative) ring with identity and let $\mathfrak{a}$ be the ideal of $R[g_2, g_3, X, Y, Z]$ generated by the homogenized equation

$$Y^2 Z = 4X^3 - g_2 X Z^2 - g_3 Z^3$$

of (2.1), i.e., $\mathfrak{a} := \langle -Y^2 Z + 4X^3 - g_2 X Z^2 - g_3 Z^3 \rangle$. Then the *Weierstrass family* is the algebraic family over the Euclidean two-space over R, $\operatorname{Spec} R[g_2, g_3] =: \mathbb{A}^2(R)$:

$$\mathbb{W}_R := \operatorname{Proj}(R[g_2, g_3, X, Y, Z]/\mathfrak{a}), \tag{2.4}$$

where $R[g_2, g_3, X, Y, Z]$ is considered as the graded $R[g_2, g_3]$-algebra such that each of X, Y, Z has degree +1 and the elements of $R[g_2, g_3]$ all have degree zero.

Also, put $\mathfrak{b} := \langle -Y^2 Z + aX^3 + bX^2 Z + cXZ^2 + dZ^3 \rangle$ and consider the algebraic family over

$$\operatorname{Spec}(R[a, a^{-1}, b, c, d]) = \mathbb{A}^4(R) - \{\text{the hypersurface } a = 0\} :$$

$$\mathbb{G}_R := \operatorname{Proj}(R[a, a^{-1}, b, c, d, X, Y, Z]/\mathfrak{b}). \tag{2.5}$$

If 2 is invertible in R, then the base $\operatorname{Spec}(R[g_2, g_3])$ of the Weierstrass family $\mathbb{W}_R$ is a closed subscheme of the base $\operatorname{Spec}(R[a, a^{-1}, b, c, d])$ of the algebraic family $\mathbb{G}_R$ of (2.5). Namely, $\operatorname{Spec}(R[g_2, g_3])$ is the closed subscheme defined by the ideal

$$\langle a - 4, b, c + g_2, d + g_3 \rangle.$$

Note that the Weierstrass family $\mathbb{W}_R$ is the pull-back of $\mathbb{G}_R$ under the closed immersion

$$\mathrm{Spec}(R[g_2, g_3]) \hookrightarrow \mathrm{Spec}(R[a, a^{-1}, b, c, d]). \tag{2.6}$$

On the other hand, we have observed earlier the following: when 2 and 3 are invertible in R, the general cubic equation (2.2) can be reduced to the Weierstrass equation (2.1). That is, if 6 is invertible in R, we have found an R-algebra homomorphism

$$R[g_2, g_3] \to R[a, a^{-1}, b, c, d], \tag{2.7a}$$

or a morphism of affine schemes over $\mathrm{Spec}\, R$,

$$\mathrm{Spec}(R[a, a^{-1}, b, c, d]) \to \mathrm{Spec}(R[g_2, g_3]) \tag{2.7b}$$

so that the pull-back of the Weierstrass family $\mathbb{W}_R$ under (2.7b) is canonically isomorphic to $\mathbb{G}_R$.

Let us apply the Jacobian Criterion to the Weierstrass affine algebraic family defined by

$$\mathrm{Spec}\left(R[g_2, g_3, X, Y]/\langle -Y^2 + 4X^3 - g_2 X - g_3\rangle\right) \tag{2.8}$$

over $\mathbb{A}^2(R) = \mathrm{Spec}(R[g_2, g_3])$ to find the set of points in the base space $\mathrm{Spec}(R[g_2, g_3])$ over which the fibre is singular. For a point $\mathfrak{p} := (g_2', g_3')$ in the Euclidean 2-space $\mathbb{A}^2(R) = \mathrm{Spec}(R[g_2, g_3])$, a point (x, y) in the Weierstrass affine family (2.8) over the point (g_2', g_3') is a singular point in the fibre if and only if the Jacobian Criterion holds: the polynomials

$$\begin{cases} \dfrac{\partial}{\partial Y}(-Y^2 + 4X^3 - g_2' X - g_3') \\ \dfrac{\partial}{\partial X}(-Y^2 + 4X^3 - g_2' X - g_3') \end{cases} \tag{2.9a}$$

in $\mathbb{k}(\mathfrak{p})[X, Y]$ vanish at the point (x, y) in the fibre of the Weierstrass affine family (2.8), where $\mathbb{k}(\mathfrak{p})$ is the residue class field at $\mathfrak{p} = (g_2', g_3')$. Therefore, a singular point in the fibre must satisfy

$$\begin{cases} -2Y = 0 \\ 12X^2 - g_2' = 0 \end{cases} \tag{2.9b}$$

and

$$-Y^2 + 4X^3 - g_2' X - g_3' = 0. \tag{2.10}$$

That is, in order to find all the points (g_2', g_3') in the base where the fibres of (2.8) are singular, we must find all the points $\mathfrak{p} \in \mathbb{A}^2(R)$ to satisfy (2.9b) and (2.10). Such a solution exists if and only if

$$\begin{cases} 12x^2 = g_2' \\ 4x^3 - g_2' x - g_3' = 0 \end{cases} \tag{2.11}$$

have a simultaneous solution in a universal domain K of $\kappa := \Bbbk(\mathfrak{p})$. Such a K can be any fixed algebraically closed field of infinite transcendental degree over $\kappa = \Bbbk(\mathfrak{p})$. From the first equation in (2.11), the only solutions are

$$x = \pm\frac{1}{2}\sqrt{\frac{g_2'}{3}}. \tag{2.12a}$$

In order to satisfy the second equation of (2.11), we must have

$$\pm 4\left(\frac{1}{2}\sqrt{\frac{g_2'}{3}}\right)^3 \mp \frac{g_2'}{2}\sqrt{\frac{g_2'}{3}} - g_3' = 0 \tag{2.12b}$$

in K. Namely, we have

$$\mp\frac{1}{3}\left(\frac{g_2'}{3}\right)^{\frac{3}{2}} = g_3' \tag{2.12c}$$

Equation (2.12c) has a solution if and only if the square has a solution, i.e.,

$$\left(g_2'\right)^3 - 27\left(g_3'\right)^2 = 0. \tag{2.12d}$$

Therefore the fibre over a point $\mathfrak{p} \in \mathbb{A}^2(R)$ of the affine Weierstrass family (2.8) has a non-simple point if and only if $\mathfrak{p}$ is on the hypersurface

$$\Delta := g_2^3 - 27g_3^2 \tag{2.13}$$

in $\mathbb{A}^2(R)$. That is, the fibre over $\mathfrak{p} \in \mathbb{A}^2(R) = \operatorname{Spec}(R[g_2, g_3])$ contains a non-simple point if and only if $\mathfrak{p}$ is on the closed subscheme of $\mathbb{A}^2(R) = \operatorname{Spec}(R[g_2, g_3])$ defined by the ideal in $R[g_2, g_3]$ generated by the element $\Delta = g_2^3 - 27g_3^2$.

Let $\mathfrak{p}$ be on the hypersurface defined by $\Delta = g_2^3 - g_3^2 = 0$ in $\mathbb{A}^2(R)$ (i.e., the images of g_2' and g_3' of g_2 and g_3 in the field κ satisfy $\left(g_2'\right)^3 - 27\left(g_3'\right)^2 = 0$ in κ). We shall see how many singular points there are in the fibre over $\mathfrak{p}$. We have observed that a point (x, y), $x, y \in K$, is a singular point if and only if (x, y) satisfies equations (2.9b) and (2.10) in K, i.e., $y = 0$ and equations (2.12a) and (2.12c) hold. If $g_3' = 0$ then (2.12d) implies $g_2' = 0$. For $g_2' = 0$, from (2.12a), $x = 0$. Namely, we have a unique singular rational point $X = Y = 0$ in the fibre over $\mathfrak{p}$. If $g_3' \neq 0$, then there exists only one solution (x, y) satisfying (2.9b) and (2.10). That is, $y = 0$ and x equals either

$$\frac{1}{2}\sqrt{\frac{g_2'}{3}} \qquad \text{or} \qquad -\frac{1}{2}\sqrt{\frac{g_2'}{3}}.$$

Note also that from (2.3) we get

$$x^2 = \frac{g_2'}{12}. \tag{2.14}$$

From equation (2.12c) we conclude $y = 0$. Then $4x^3 - g_2'x - g_3' = 0$. Substituting $x^2 = g_2'$ of (2.14) into $x(4x^2 - g_2') - g_3' = 0$, we get

$$x\Big(\frac{4g_2'}{12} - g_2'\Big) - g_3' = 0. \tag{2.15}$$

Namely, assuming $g_2' \neq 0$, we have

$$x = -\frac{3}{2}\frac{g_3'}{g_2'}. \tag{2.16}$$

When $g_2' = 0$, $\big(g_2'\big)^3 - 27\big(g_3'\big)^2 = 0$ implies that $g_3' = 0$. This case has already been considered.

Summarizing the above discussion, for a point $\mathfrak{p} \in \mathrm{Spec}(R[g_2, g_3])$ the fibre of the affine family (2.8) over $\kappa = \Bbbk(\mathfrak{p})$ is singular if and only if the images g_2' and g_3' of g_2 and g_3 in κ under the natural epimorphism $R[g_2, g_3] \twoheadrightarrow \kappa$ satisfy (2.12d). Then the fibre $\mathrm{Spec}\big(\kappa[X,Y]\big)/\langle -Y^2 + 4X^3 - g_2X - g_3\rangle\big)$ has a unique non-simple point. This non-simple point is a κ-rational point given by

$$(x, y) = \Big(-\frac{3}{2}\frac{g_3'}{g_2'},\ 0\Big) \tag{2.17}$$

for $g_2' \neq 0$ (then $g_3' \neq 0$). In the case where $g_2' = g_3' = 0$, the rational point is $(0, 0)$.

The affine family (2.8), i.e.,

$$\mathrm{Spec}\ \big(R[g_2, g_3, X, Y]/\langle -Y^2 + 4X^3 - g_2X - g_3\rangle\big)$$

is the open family defined by "$Z \neq 0$" or "$Z = 1$" in the Weierstrass projective family $\mathbb{W}_R$ over $\mathbb{A}^2(R)$. Let (x, y, z) be a point on $\mathbb{W}_R$ satisfying the homogeneous equation

$$-Y^2Z + 4X^3 - g_2'XZ^2 - g_3'Z^3, \tag{2.18}$$

where (g_2', g_3') corresponds to a point in $\mathbb{A}^2(R)$ as mentioned before, and (x, y, z) is a set of homogeneous coordinates in a universal domain K for $\kappa = \Bbbk(\mathfrak{p})$. For (x, y, z) satisfying (2.18), if $z = 0$ then $x = 0$. Therefore, the affine open "$X \neq 0$" of $\mathbb{P}^2(R)$, the projective 2-space, meets the Weierstrass family $\mathbb{W}_R$ in a subset of the affine open "$Z \neq 0$". Therefore, there are some points of $\mathbb{W}_R$ which are not on the affine family (2.8), i.e., on the open "$Z \neq 0$" of $\mathbb{P}^2(R)$ and on the affine "$X \neq 0$". Therefore, such a new point on $\mathbb{W}_R$ that is not on the affine "$Z \neq 0$" must be on the affine open "$Y \neq 0$". The intersection between $\mathbb{W}_R$ and "$Y \neq 0$" can be obtained by letting $Y = 1$ in (2.18)

$$-Z + 4X^3 - g_2XZ^2 - g_3Z^3. \tag{2.19}$$

By the Jacobian Criterion, a point (x, z) on the affine open (2.19) of $\mathbb{W}_R$ over (g_2', g_3') is a non-simple point in the fibre if and only if the following polynomials with coefficients in $\kappa = \Bbbk(\mathfrak{p})$:

$$\begin{cases} \frac{\partial}{\partial Z}(-Z + 4X^3 - g_2'XZ^2 - g_3'Z^3) \\ \frac{\partial}{\partial X}(-Z + 4X^3 - g_2'XZ^2 - g_3'Z^3) \end{cases} \tag{2.20a}$$

vanish at (x, z). That is,

$$\begin{cases} -1 - 2g_2'XZ - 3g_3'Z^2 = 0 \\ 12X^2 - g_2'Z^2 = 0. \end{cases} \tag{2.20b}$$

Since we have covered all the non-simple points in the fibres which are on the affine open "$Z \neq 0$" of $\mathbb{W}_R$ we will study those points of $\mathbb{W}_R$ which are not on the affine open (2.8):

$$Z = 0.$$

Notice that the equations in (2.20b) have no solutions in any fibre of $\mathbb{W}_R$ over any point (g_2', g_3') in $\mathbb{A}^2(R)$. This is because $z = 0$ implies $x = 0$ from the second equation of (2.20b). Then the first equation becomes $-1 - 2g_2' \cdot 0 - 3g_3 \cdot 0 = 0$, i.e., $1 = 0$. Namely, every point of $\mathbb{W}_R$ which is a non-simple point in the fibre over $\mathbb{A}^2(R)$ is on the affine open (2.20b). We shall call those points of $\mathbb{W}_R$ that are not in the affine open "$Z \neq 0$", i.e., (2.8), the "points at ∞ of $\mathbb{W}_R$". What we have observed in the above can be rephrased as *all the points at ∞ of $\mathbb{W}_R$ are simple points in each fibre*. That is, as for singularities, we only need to pay attention to the "finite points" on the affine family (2.8).

Let us observe that the Weierstrass family $\mathbb{W}_R$ is the closed subscheme of $\mathbb{P}^2\big(\mathrm{Spec}(R[g_2, g_3])\big)$ determined by $Y^2Z = 4X^2g_2XZ^2 - g_3Z^3$. On the other hand, the Weierstrass affine open family (2.8), i.e., "$Z \neq 0$" is the closed subset of $\mathbb{A}^2\big(\mathrm{Spec}(R[g_2, g_3])\big)$ given by $Y^2 = 4X^3 - g_2X - g_3$. The closed subset of the points at ∞ of $\mathbb{W}_R$, i.e., "$Z = 0$", is the complement of the affine open "$Z \neq 0$". For homogeneous coordinates (x, y, z), where x, y and z are in a universal domain K for κ as before, (x, y, z) is a point on $\mathbb{W}_R$ if and only if (x, y, z) satisfies the homogeneous equation

$$-Y^2Z + 4X^3 - g_2'XZ^2 - g_3'Z^3 = 0$$

in K. Then (x, y, z) is a point at ∞ if and only if (x, y, z) satisfies $Z = 0$. That is, if the point of $\mathbb{P}^2(\mathrm{Spec}\,\kappa)$ defined by (x, y, z) is a point at ∞ of $\mathbb{W}_R$ over $\mathfrak{p} \in \mathrm{Spec}(R[g_2, g_3])$, then $z = 0$. By the equation

$$-Y^2Z + 4X^3 - g_2'XZ^2 - g_3'Z^3 = 0$$

we then have $x = 0$. Therefore, for every $\mathfrak{p}$ there is one and only one point at ∞ of $\mathbb{W}_R \subset \mathbb{P}^2\big(\mathrm{Spec}(R[g_2, g_3])\big)$ in the fibre of $\mathbb{W}_R$ over $\mathfrak{p}$. This point is the point of the fibre $\mathbb{P}^2(\operatorname{Spec}\kappa)$ of $\mathbb{P}^2\big(\operatorname{Spec}(R[g_2, g_3])\big)$ over $\mathfrak{p} \in \operatorname{Spec}(R[g_2, g_3])$, given by the homogeneous coordinates $(0, 1, 0)$. Consequently, the unique point at ∞ of $\mathbb{W}_R$ over $\Bbbk(\mathfrak{p})$ is a rational point for every $\mathfrak{p}$ in the base $\operatorname{Spec}(R[g_2, g_3])$. Note also that the closed subscheme of points at infinity of $\mathbb{W}_R$ over $\mathbb{A}^2(R)$ is isomorphic to $\mathbb{A}^2(R)$ over $\mathbb{A}^2(R)$, and that the closed subscheme is contained in the set of simple points of $\mathbb{W}_R$ over $\mathbb{A}^2(R)$.

Summarizing the above discussion, there is one and only one point at infinity in each fibre. Such a unique point at ∞ in each fibre is a simple point and a κ-rational point in the fibre over $\mathfrak{p}$ for every $\mathfrak{p} \in \mathbb{A}^2(R)$. Namely, all the singular phenomena occur on the affine open ($Z \neq 0$), i.e., on the Weierstrass affine family (2.8).

We will study singular fibres next. We have previously observed that for $\mathfrak{p} \in \operatorname{Spec}(R[g_2, g_3])$, the fibre of $\mathbb{W}_R$ over $\mathfrak{p}$ is singular if and only if $\mathfrak{p}$ is on the closed subscheme

$$g_2^3 - 27g_3^2 = 0 \tag{2.21}$$

of the above base scheme. Or one can say that the images g_2' and g_3' of g_2 and g_3 in $\kappa = \Bbbk(\mathfrak{p})$ satisfy (2.21). Recall that for a polynomial P of degree n over a field, the *discriminant* of P is defined by

$$\Delta := \prod_{\substack{1 \leq i,j \leq n \\ i \neq j}} (\varrho_i - \varrho_j), \tag{2.22}$$

where $\varrho_1, \varrho_2, \ldots, \varrho_n$ are the roots of P in an algebraic closure. Then the three roots of the polynomial $4X^3 - g_2X - g_3$ are not distinct if and only if equation (2.21) holds. That is, the affine curve over the field κ

$$Y^2 = 4X^3 - g_2X - g_3$$

is non-singular if and only if $4X^3 - g_2X - g_3$ has three distinct roots, i.e., is a separable polynomial.

5.2.1 Singular Fibres in the Weierstrass Family

In Section 5.2 we observed that for $\mathfrak{p} \in \operatorname{Spec}(R[g_2, g_3])$ the fibre of the Weierstrass family $\mathbb{W}_R$ corresponding to R is singular if and only if $\Delta = g_2^3 - 27g_3^2$ vanishes at $\mathfrak{p}$. Furthermore, all the singular points are on the affine open

$$\operatorname{Spec}\big(\kappa[X, Y]/\langle Y^2 - 4X^3 + g_2X - g_3\rangle\big),$$

where $\kappa = \Bbbk(\mathfrak{p})$. We also observed that on this affine open, there exists a unique singular rational point over $\kappa = \Bbbk(\mathfrak{p})$. Namely, for each point $\mathfrak{p}$ on the closed subscheme $\Delta = g_2^3 - 27g_3^2 = 0$ of the base affine scheme $\mathbb{A}^2(R) = \operatorname{Spec}(R[g_2, g_3])$ for $\mathbb{W}_R$, there exists exactly one non-simple κ-rational point on the fibre of $\mathbb{W}_R$ over $\mathfrak{p}$. There are two types of singular fibres. Namely, the first type are singular fibres over points satisfying $\Delta = g_2^3 - 27g_3^2 = 0$ but $g_2 \neq 0$ (hence $g_3 \neq 0$), and the second type are fibres over points $g_2 = 0$ (hence $g_3 = 0$). Let $\mathfrak{p} \in \mathrm{Spec}(R[g_2, g_3])$ satisfying $g_2' = g_3' = 0$ in κ. Then the fibre of $\mathbb{W}_R$ over κ is given by

$$Y^2 Z = 4X^3, \tag{2.23}$$

is an equation of a cusp. If we let W be the fibre of $\mathbb{W}_R$ over $\mathfrak{p}$, then W is birationally equivalent to $\mathbb{P}(\kappa)$. Note that the fibre W over $\mathfrak{p}$ has only one singular point, called the cusp point, $x = y = 0$. Next, consider the first type, i.e.,

$$\left(g_2'\right)^3 - 27\left(g_3'\right)^2 = 0 \tag{2.24}$$

and $g_2' \neq 0$ and $g_3' \neq 0$. Consider the following affine curve over the a field k

$$Y^2 = 4(X - r_1)(X - r_2)(X - r_3). \tag{2.25}$$

Then by the Jacobian Criterion if all r_1, r_2, r_3 are distinct, the affine curve (2.25) is simple. Equation (2.23) corresponds to the case when all the roots r_1, r_2, r_3 are equal. When $4X^3 - g_2'X - g_3'$ is factored linearly in an algebraic closure $\bar{\kappa}$ of κ, only two of their roots r_1, r_2, r_3 in $\bar{\kappa}$ are equal. We have observed that such a double root of $4X^3 - g_2X - g_3$ is the unique singular point of $Y^2 = 4X^3 - g_2'X - g_3'$ given by $(x, y) = (-\frac{3}{2}\frac{g_3'}{g_2'}, 0)$. Let r be the third root of the cubic equation:

$$4X^3 - g_2'X - g_3' = 4\big(X + \frac{3}{2}\frac{g_3'}{g_2'}\big)^2(X - r). \tag{2.26}$$

The constant terms of (2.26) give $-g_3' = 4\big(g_2'X - g_3'\big)^2(-r)$, i.e.,

$$r = \frac{1}{9}\frac{\left(g_2'\right)^2}{g_3'}.$$

That is, $4X^3 - g_2'X - g_3'$ can be factored linearly over κ as

$$4X^3 - g_2X - g_3 = 4\big(X + \frac{3}{2}\frac{g_3'}{g_2'}\big)^2\big(X - \frac{1}{9}\frac{\left(g_2'\right)^2}{g_3'}\big).$$

Let $X_0 := X + \frac{3}{2}\frac{g_3'}{g_2'}Z$. Then the fibre of $\mathbb{W}_R$ over κ is of the form

$$Y^2 Z = 4X_0^2(X_0 - cZ),$$

where $0 \neq c \in \kappa$. Furthermore let $X_0 := cX_1$. Then we get

$$Y^2 Z = 4c^3 X_1^2(X_1 - Z),$$

or

$$Y^2Z = bX_1^2(X_1 - Z), \tag{2.27}$$

where $b := 4c^3$. The unique singular point in the homogeneous coordinates (X_1, Y, Z) on the projective curve (2.27) is $(0, 0, 1)$. A cubic equation as in (2.27) is said to be a projective line with an ordinary double point. If $\kappa = \Bbbk(\mathfrak{p}) = \mathbb{C}$, the field of complex numbers, the the classical singular homology of the fibre W of $\mathbb{W}_R$ over $\kappa = \mathbb{C}$ becomes

$$\mathrm{H}_1(W, \mathbb{C}) \approx \mathbb{C}.$$

For elemental properties of projective geometry, see

* Hartshorne, R., *Foundations of Projective Geometry*, Benjamin, 1967

is recommended. For elliptic curves, for example, see

* Silverman, J.H. and Tate, J., *Rational Points on Elliptic Curves*, Undergraduate Texts in Mathematics, Springer-Verlag, 1992.

5.2.2 Lifted p-adic Homology with Compact Supports of Fibres of the Weierstrass Family; The case of varieties over $\mathbb{C}$

The main reference for Chapter V is:

[LuHC] Lubkin, S., *Finite Generation of p-Adic Homology with Compact Supports. Generalization of the Weil Conjectures to Singular, Non-complete Algebraic Varieties*, Journal of Number Theory, **11**, (1979), 412–464.

Let X be a complex algebraic variety which is embeddable over $\mathbb{C}$ and let X_{top} be the closed points of X with the classical topology. Let Y be non-singular over $\mathbb{C}$ of dimension N so that X may be closed in Y. Then the definition of the homology of X with compact supports $\mathrm{H}_j^c(X, \mathbb{C})$ is the relative hypercohomology

$$\mathrm{H}_j^c(X, \mathbb{C}) := \mathrm{H}^{2N-j}(Y, Y - X, \Omega_{\mathbb{C}}^{\bullet}).$$

See [LuHC] in the above. Then since Y is non-singular over $\mathbb{C}$, we have the canonical isomorphism from $\mathrm{H}^{2N-j}(Y, Y - X, \Omega_{\mathbb{C}}^{\bullet})$ to the classical singular cohomology $\mathrm{H}^{2N-j}(Y_{\text{top}}, Y_{\text{top}} - X_{\text{top}}, \mathbb{C})$. By applying the Lefschetz duality to the oriented $2N$-dimensional manifold Y_{top} and the subspace X_{top}, we have

$$\mathrm{H}_{2N-j}(Y_{\text{top}}, Y_{\text{top}} - X_{\text{top}}, \mathbb{C}) \approx \check{\mathrm{H}}_c^j(X_{\text{top}}, \mathbb{C}) \tag{2.28}$$

where the right hand-side of (2.28) is the classical Čech cohomology. Since X is an algebraic variety, we have

$$\check{\mathrm{H}}_c^j(X_{\text{top}}, \mathbb{C}) \approx \mathrm{H}_c^j(X_{\text{top}}, \mathbb{C}). \tag{2.29}$$

Since all these cohomology groups are finitely generated over $\mathbb{C}$, passing to duality over $\mathbb{C}$, we obtain,

$$\mathrm{H}^{2N-j}(Y_{\text{top}}, Y_{\text{top}} - X_{\text{top}}, \mathbb{C}) \approx \mathrm{H}^c_j(X_{\text{top}}, \mathbb{C}). \tag{2.30}$$

That is, we have $\mathrm{H}^c_j(X, \mathbb{C}) \approx \mathrm{H}^c_j(X_{\text{top}}, \mathbb{C})$.

In particular, if X is an embeddable complete complex algebraic variety, then we get $\mathrm{H}^c_j(X, \mathbb{C}) \approx \mathrm{H}_j(X_{\text{top}}, \mathbb{C})$, the classical singular homology. This is because singular homology with compact supports is the same as ordinary singular homology. When X is a fibre of the Weierstrass family over $\mathfrak{p}$ where $\Bbbk(\mathfrak{p}) = \mathbb{C}$, then $\mathrm{H}^c_j(X, \mathbb{C})$ is isomorphic to the usual singular homology of X with complex coefficient. Namely, we have

$$\begin{cases} \mathrm{H}^c_0(X,\mathbb{C}) \approx \mathrm{H}^c_2(X,\mathbb{C}) \approx \mathbb{C}, \\ \mathrm{H}^c_1(X,\mathbb{C}) \approx \begin{cases} \mathbb{C} \oplus \mathbb{C}, & \text{for an elliptic curve} X \\ \mathbb{C}, & \text{for a projective line with ordinary double point} \\ 0, & \text{for a projective line with a cusp,} \end{cases} \\ \mathrm{H}^c_j(X,\mathbb{C}) = 0, \text{for } j \neq 0,1,2. \end{cases} \tag{2.31}$$

Next, let us consider varieties over characteristic zero fields. Let K be a field of characteristic zero and let L be an extension field of K. For an algebraic variety X over K which is embeddable over K, $X \times_K L$ is an algebraic variety over L and is embeddable over L. Let Y be non-singular over K containing X as a closed subvariety. Then $Y \times_K L$ contains $X \times_K L$ as a closed subvariety over L. Then $Y \times_K L$ is affine over Y and the direct image of $\Omega^\bullet_L(Y \times_K L)$ is $\Omega^\bullet_K(Y) \otimes_K L$. Therefore, we have an isomorphism

$$\begin{aligned} &\mathrm{H}^j(Y, Y - X, \Omega^\bullet_K(Y)) \otimes_K L \approx \\ &\qquad \approx \mathrm{H}^j(Y \times_K L, (Y \times_K L) - (X \times_K L), \Omega^\bullet_L(Y \times_K L)), \end{aligned} \tag{2.32}$$

namely,

$$\mathrm{H}^c_j(X, K) \otimes_K L \approx \mathrm{H}^c_j(X \times_K L, L) \tag{2.33}$$

as vector spaces over L for all j. In the constant characteristic zero one can generalize the above as follows. Let K and L be rings containing the field $\mathbb{Q}$ of rational numbers. For a ring homomorphism from K to L, we have a right-half-plane spectral sequence

$$\mathrm{Tor}_p(\mathrm{H}^c_q(X, K), L)$$

abutting upon $\mathrm{H}^c_n(X, L)$.

When K can be embedded in $\mathbb{C}$, (2.32) implies

$$\mathrm{H}^c_j(X, K) \otimes_K \mathbb{C} \approx \mathrm{H}^c_j(X \times_K \mathbb{C}, \mathbb{C}). \tag{2.34}$$

We have shown that the right hand-side $\mathrm{H}^c_j(X \times_K \mathbb{C}, \mathbb{C})$ is the classical complex homology with compact supports of the complex variety $X \times_K \mathbb{C}$. If X is complete, then it is the classical complex homology of $X \times_K \mathbb{C}$. In the case where K is an arbitrary field of characteristic zero, let K_0 be a subfield of K which is finitely generated over $\mathbb{Q}$ so that there may exist an embeddable algebraic variety X_0 over K_0 satisfying $X_0 \times_{K_0} K \approx X$ as varieties over K. Then, by (2.33), we get

$$\mathrm{H}^c_j(X, K) \approx \mathrm{H}^c_j(X_0, K_0) \otimes_{K_0} K.$$

For $\mathrm{H}^c_j(X_0, K_0)$ we can use (2.34).

Such a method of reducing characteristic zero varieties to the case of varieties over $\mathbb{C}$ is called the *Lefschetz principle*.

We next consider homologies with compact supports of fibres of the Weierstrass family in characteristic zero. From what we have discussed in the above, we obtain the following: Let R be a commutative ring with identity and let X be a fibre of the Weierstrass family $\mathbb{W}_R$ at $\mathfrak{p} \in \mathrm{Spec}(R[g_2, g_3])$ where the characteristic of $\kappa = \Bbbk(\mathfrak{p})$ is zero. Then we have

$$\begin{cases} \mathrm{H}^c_0(X, \kappa) \approx \mathrm{H}^c_2(X, \kappa) \approx \kappa, \\ \mathrm{H}^c_1(X, \kappa) \approx \begin{cases} \kappa \oplus \kappa, & \text{if } X \text{ is an elliptic curve} \\ & \text{(namely, } X \text{ is a non-singular fibre)} \\ \kappa, & \text{for a projective line with} \\ & \text{ordinary double point} \\ 0, & \text{if } X \text{ is a projective line with a cusp,} \end{cases} \\ \mathrm{H}^c_j(X, \kappa) = 0, \text{for } j \neq 0, 1, 2. \end{cases} \tag{2.35}$$

Let us now consider the case where the characteristic of the field $\Bbbk(\mathfrak{p})$ at $\mathfrak{p} \in \mathrm{Spec}(R[g_2, g_3])$ is $p \neq 0$. As before, let X be a fibre of $\mathbb{W}_R$ over $\mathfrak{p}$ in the base. Let $\mathscr{O}$ be a complete discrete valuation ring with mixed characteristics having k as the residue class field and K as the quotient field of $\mathscr{O}$. Then for a non-singular and proper lifting X_K over K of the fibre X over k (i.e., X_K is an elliptic curve), we have

$$\mathrm{H}^c_j(X, K) = \mathrm{H}^{2N-j}(X, K)$$

by taking $Y = X$. The right hand-side is the lifted p-adic cohomology in

[LuPWC] Lubkin, S., *A p-Adic Proof of Weil's Conjectures*, Ann. of Math. (2) **87**, (1968), 105–255,

and $\mathrm{H}^{2N-j}(X, K)$ is the hypercohomology $\mathrm{H}^{2N-j}(X_K, K)$ in [LuPWC]. That is, we obtain $\mathrm{H}^c_j(X, K) \approx \mathrm{H}^c_j(X_K, K)$. Therefore, we get

$$\begin{cases} \mathrm{H}^c_0(X, K) \approx \mathrm{H}^c_2(X, K) \approx K \\ \mathrm{H}^c_1(X, K) \approx K \oplus K & \text{if } X \text{ is non-singular.} \end{cases}$$

If X is a singular fibre, by direct computation we obtain

$$\mathrm{H}^c_j(X, K) \approx \begin{cases} K & \text{for } j = 0, 2 \\ 0 & \text{for } j \neq 0, 2 \end{cases}$$

$$\mathrm{H}^c_1(X, K) \approx \begin{cases} K & \text{if } X \text{ is a projective line with} \\ & \text{an ordinary double point} \\ 0 & \text{if } X \text{ is projective line with a cusp.} \end{cases}$$

5.2.3 The Universal Coefficient Spectral Sequence

Let $\mathscr{O}$ be a complete discrete valuation ring with the quotient field of characteristic zero and the residue class field k. (If $\mathscr{O}$ is a field then $K = \mathscr{O} = k$.) Then we have the following spectral sequence in [LuHC] especially (26) on page 426.

Theorem 18. *Let $\underline{A}$ be an $\mathscr{O}$-algebra and let $\underline{B}$ be an $\underline{A}$-algebra. We also let $A := (\underline{A} \otimes_{\mathscr{O}} k)_{red}$ and $B := (\underline{B} \otimes_{\mathscr{O}} k)_{red}$. For a scheme X over A which is embeddable over $\underline{A}$, let $X_B := X \times_{\mathrm{Spec}(A)} \mathrm{Spec}(B)$. Then there exists a right-half-plane homological spectral sequence called the* universal coefficient spectral sequence*:*

$$E^2_{p,q} := \mathrm{Tor}_p^{\underline{A}^\dagger \otimes_{\mathscr{O}} K}(\mathrm{H}^c_q(X, \underline{A}^\dagger \otimes_{\mathscr{O}} K), \underline{B}^\dagger \otimes_{\mathscr{O}} K) \tag{2.36}$$

abutting to $\mathrm{H}^c_n(X_B, \underline{B}^\dagger \otimes_{\mathscr{O}} K)$.

We are interested in a special case of the spectral sequence (2.36) as follows. Let $F : \underline{A} \to \underline{A}$ be an endomorphism of $\underline{A}$ so that F induces the p-th power endomorphism of $A = \underline{A}/p\underline{A}$, where p is the characteristic of $k = \Bbbk(\mathscr{O})$. Then there is a unique ring homomorphism

$$\underline{A} \longrightarrow W(A), \tag{2.37}$$

where $W(A)$ is the Witt vector on $A = \underline{A}/p\underline{A}$. See

[LuBW] Lubkin, S., *Generalization of p-Adic Cohomology; Bounded Witt vectors*, Compositio Math. **34**, (1977)

for Witt vector cohomology[1], and for the †-completion, see [LuPWC]. Let $\mathfrak{p} \in \operatorname{Spec}(A)$, where $\Bbbk(\mathfrak{p})$ is a perfect field. Then there exists a natural homomorphism from $\underline{A}$ to the Witt vector on $\Bbbk(\mathfrak{p})$, i.e.,

$$\underline{A} \longrightarrow W(\Bbbk(\mathfrak{p}))$$

where $W(\Bbbk(\mathfrak{p}))$ is the unique mixed characteristic complete discrete valuation ring having $\Bbbk(\mathfrak{p})$ as its residue class field. For our Weierstrass family case, we let $\underline{A} := \hat{\mathbb{Z}}_p[g_2, g_3]$. For a maximal ideal $\mathfrak{p} \in \operatorname{Spec}(A)$ (i.e., a closed point), $\Bbbk(\mathfrak{p})$ is a finite field. Let g_2' and g_3' be the images of g_2 and g_3 in $\Bbbk(\mathfrak{p})$. We can construct the Witt vector $W(\Bbbk(\mathfrak{p}))$ as follows. Each if g_2' and g_3' is either a root of unity of order prime to p or else zero. Let ϱ be an element of $\Bbbk(\mathfrak{p})$ which is a multiplicative generator of the cyclic group $\Bbbk(\mathfrak{p}) - \{0\}$. Then each element of $\Bbbk(\mathfrak{p})$, including g_2' and g_3', is either a power of ϱ or else zero. Let a be the multiplicative order of ϱ. Embed $\hat{\mathbb{Z}}_p$ as a subring of $\mathbb{C}$ and let $'\varrho$ be any fixed root of unity in $\mathbb{C}$ of order exactly a. Then the subring generated by $\hat{\mathbb{Z}}_p$ and $'\varrho$ in $\mathbb{C}$ is the Witt vector $W(\Bbbk(\mathfrak{p})) = \hat{\mathbb{Z}}_p['\varrho]$. For $g_2' = \varrho^i$, let $'g_2' = ('\varrho)^i$ (and similarly for $'g_3'$). For $g_2' = 0$, let $'g_2' = 0$.

Our special case of the universal coefficient spectral sequence is obtained as follows. As before, let $\underline{A}$ be an $\mathscr{O}$-algebra and let F be any ring endomorphism of $\underline{A}$ so that F may induce the p-th power endomorphism of $\underline{A}/p\underline{A}$. For any prime ideal of $\operatorname{Spec}(\underline{A} \otimes_{\mathscr{O}} k)_{\text{red}} = \operatorname{Spec}(\underline{A}/p\underline{A}) = \operatorname{Spec}(A)$, we get a natural homomorphism $\underline{A} \to W(\Bbbk(\mathfrak{p}))$ as in the above. For our case, $\underline{A} := \hat{\mathbb{Z}}_p[g_2, g_3]$, and $\mathfrak{p}$ is a maximal ideal of A. Then we have $\Bbbk(\mathfrak{p}) = (\mathbb{Z}/p\mathbb{Z})[g_2', g_3']$ and $W(\Bbbk(\mathfrak{p})) = \hat{\mathbb{Z}}_p['g_2', 'g_3']$. The natural homomorphism $\underline{A} \to W(\Bbbk(\mathfrak{p}))$ becomes

$$\hat{\mathbb{Z}}_p[g_2, g_3] \longrightarrow \hat{\mathbb{Z}}_p['g_2', 'g_3']$$

defined by $g_2 \mapsto 'g_2'$ and $g_3 \mapsto 'g_3'$. Let $\underline{B} = W\big(\Bbbk(\mathfrak{p})^{p^{-\infty}}\big)$ where $\Bbbk(\mathfrak{p})^{p^{-\infty}}$ is the purely inseparable algebraic closure of $\Bbbk(\mathfrak{p})$. Then $\underline{B} = W\big(\Bbbk(\mathfrak{p})^{p^{-\infty}}\big)$ is a complete discrete valuation ring of mixed characteristic having $\Bbbk(\mathfrak{p})^{p^{-\infty}}$ as the residue class field, and $\underline{B} \otimes_{\mathbb{Z}} \mathbb{Q}$ is a field of characteristic zero. For a scheme X over $\operatorname{Spec}(A)$ which is embeddable over $\underline{A}$, the fibre $X_{\mathfrak{p}}$ over $\Bbbk(\mathfrak{p})$ is an algebraic variety over the field $\Bbbk(\mathfrak{p})$. Then let $Y_{\mathfrak{p}} := X_{\mathfrak{p}} \times_{\Bbbk(\mathfrak{p})} \Bbbk(\mathfrak{p})^{p^{-\infty}}$. The zeta matrices have coefficients in the quotient field $K_{\mathfrak{p}} = \underline{B} \otimes_{\mathbb{Z}} \mathbb{Q}$ of the complete discrete valuation ring $\underline{B} = W(\Bbbk(\mathfrak{p})^{p^{-\infty}})$. Then the universal

[1] Private communication with Pierre Deligne; Boundedness condition in bounded Witt cohomology is not necessary.

coefficient spectral sequence in [LuHC] becomes

$$E^2_{p,q} = \mathrm{Tor}_p^{\underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}}(\mathrm{H}^c_q(X, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q}), K_{\mathfrak{p}}) \tag{2.38}$$

abutting to $\mathrm{H}^c_n(Y_{\mathfrak{p}}, K_{\mathfrak{p}})$. Namely, the lifted p-adic homology with compact supports of the algebraic family X over $\mathrm{Spec}(A)$ computes the lifted p-adic homology with compact supports of all the fibres in the family. Furthermore, the zeta endomorphisms of $\mathrm{H}^c_q(X, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$ will compute the zeta endomorphisms of the lifted p-adic homology with compact supports of every fibre $Y_{\mathfrak{p}}$. For a finite field $\Bbbk(\mathfrak{p})$, if the $E^2_{p,q}$-term of (2.38) is a finite-dimensional vector space over $K_{\mathfrak{p}}$ for all p and q (and if $E^2_{p,q} = 0$ for all except finitely many p and q), then the zeta function of the fibre $X_{\mathfrak{p}} = Y_{\mathfrak{p}}$ is given as follows: Letting $P_{p,q}$ be the (reverse) characteristic polynomial of the endomorphism of $E^2_{p,q}$ induced by the (p^r)-power map, $p^r = \mathrm{card}(\Bbbk(\mathfrak{p}))$,

$$Z_{X_{\mathfrak{p}}}(T) = \frac{\prod_{p+q=\mathrm{odd}} P_{p,q}(T)}{\prod_{p+q=\mathrm{even}} P_{p,q}(T)}. \tag{2.39}$$

See [LuHC] for (2.39). Thus, we can compute the zeta function of every fibre over a finite field in the algebraic family of X over A. A zeta endomorphism is said to be a *zeta matrix* for a free module $\mathrm{H}^c_q(X, \underline{A}^\dagger \otimes_{\mathbb{Z}} \mathbb{Q})$.

For the explicit computation, one may be interested in the results in the following papers.

[KaLu] Kato, G. and Lubkin, S., *Zeta Matrices of Elliptic Curves*, Journal of Number Theory, **15**, No. 3, (1982), 318–330.

[KaChZ] Kato, G., *On the Generators of the First Homology with Compact Supports of the Weierstrass Family in Characteristic Zero*, Trans., AMS., **278**, (1983), 361–368.

[KaZM] Kato, G., *Liftedp-Adic Homology with Compact Supports of the Weierstrass Family and its Zeta Endomorphism*, Journal of Number Theory, **35**, No.2, (1990), 216–223.

5.2.4 Letter from Dwork

Here is the quotation from a letter written by B. Dwork[2] which may give more insight into the connection between Lubkin's p-adic cohomology and Dwork's work on p-adic analysis.

[2]In the early 1980's a few letters addressed to the author were received from Professor B. Dwork. Only the copy of this letter was sent to me (rather than the original one) where the date was cut off in the process of copying. Hence the exact date cannot be identified.

“ [. . .] I have studied the family of elliptic curves

$$Y^2 = X(1-X)(1-\lambda X)$$

of my book (Springer 1982) and in particular the appendix by Adolphson. I have also studied $X^3+Y^3+Z^3-3\Gamma XYZ = 0$ (§8 Ann. of Math. **80**, (1964), pp 227–299).

I have never made a detailed study of $Y^2 = 4X^3 - g_2X - g_3$. However the relation between those different families is well known and one can pass from the λ to j invariant and vice-versa.

If however I were to start a study of the Weierstrass family, I would suggest the following: Let $k = \mathbb{Q}(\zeta_p)$, $\zeta_p^p = 1$, $\pi^{p-1} = -p$, $\pi \in k$. Let $\mathscr{L}$ be the ring of all polynomials in $k[t, X]$ spanned by monomials $t^l X^m$ such that $3l \geq m$. Let L be the completion of $\mathscr{L}$ in the sense of series

$$\xi = \sum_{3l\geq m} A_{l,m} t^l X^m \in k[[t, X]]$$

which converge in a polydisk $|t| < 1+\epsilon$, $|X| < 1+\epsilon$.

The key to the study of

$$Y^2 = f(x)(= 4X^3 - g_2X - g_3)$$

is the operator

$$\begin{aligned} \alpha =& \psi \circ t^{\frac{p-1}{2}} F(X,t) \\ & F(X,t) = \exp\pi\big((4X^3 - g_2X - g_3)t - t^p(4X^{3p} - g_2^pX^p - g_3^p)\big) \\ & (\text{say } p \neq 2,3). \end{aligned}$$

The cohomology is given by the space

$$\mathscr{W}_{g_2,g_3} = L\big/D_1L + D_2L \cong \mathscr{L}\big/D_1\mathscr{L} + D_2\mathscr{L} \tag{2.40}$$

(where the isomorphism (2.40) is subject to conditions such as $|g_3| = |g_2| = |\Delta| = 1$, $\Delta = g_2^3 - 27g_3^2$) where

$$\begin{aligned} D_1 &= \frac{1}{t^{-\frac{1}{2}}\exp\pi t f} \circ X\frac{\partial}{\partial X} \circ t^{-\frac{1}{2}} \exp\pi t f(X) \\ D_2 &= \frac{1}{t^{-\frac{1}{2}}\exp\pi t f} \circ t\frac{\partial}{\partial t} \circ t^{-\frac{1}{2}} \exp\pi t f(X) \end{aligned}$$

i.e.,

$$\begin{aligned} D_1 &= X\frac{\partial}{\partial X} + \pi t(12X^3 - g_2X) \\ D_2 &= t\frac{\partial}{\partial t} - \frac{1}{2} + \pi t(4X^3 - g_2X - g_3). \end{aligned}$$

Furthermore,

$$\alpha : \xi \in L \longmapsto \psi\big(\xi t^{\frac{p-1}{2}} F(X,t)\big)$$

gives by passage to quotients a map of

$$\bar{\alpha} : \mathscr{W}_{g_2,g_3} \longrightarrow \mathscr{W}_{g_2,g_3}.$$

When we specialize g_2, g_3 such that $g_2 = g_2^p$, $g_3 = g_3^p$, then $\bar{\alpha}$ becomes an endomorphism, and its characteristic polynomial gives the zeta function of the reduced curve.

The differential equations of deformation are given by the actions of $\sigma_{g_2}, \sigma_{g_3}$ on $\mathscr{W}_{g_2,g_3}$

$$\sigma_{g_2} = \frac{1}{t^{-\frac{1}{2}} \exp t\pi f} \circ \frac{\partial}{\partial g_2} \circ t^{-\frac{1}{2}} \exp t\pi f$$
$$\sigma_{g_3} = \frac{1}{t^{-\frac{1}{2}} \exp t\pi f} \circ \frac{\partial}{\partial g_3} \circ t^{-\frac{1}{2}} \exp t\pi f$$

i.e.,

$$\sigma_{g_2} = \frac{\partial}{\partial g_2} - \pi t X$$
$$\sigma_{g_3} = \frac{\partial}{\partial g_3} - \pi t.$$

The matrix $\bar{\alpha}$ (defined above for $|g_2| = |g_3| = |\Delta| = 1$) is no doubt holomorphic as function of g_2, g_3 on a set of the type

$$|g_2| < 1 + \epsilon, \quad |\Delta| > 1 - \epsilon$$
$$|g_3| < 1 + \epsilon.$$

An account of this theory at the cochain level (i.e., of α but not of $\bar{\alpha}$) may be found in Adolphson's article recently published in Pacific J. Math, "On the Dwork Trace Formula". ”

Remark 18. See the following paper and references in this paper.

[Dwork] Dwork, B., *p-Adic Cycles*, Pub. Math. I.H.E.S., **37**, (1969), 27–116.

Recent works of K.S. Kedlaya on zeta function computation through p-adic cohomology can be found in

[Ked] Kedlaya, K.S., *Counting Points on Hyperelliptic Curves using Monsky–Washnitzer Cohomology*, Journal of the Ramanujan Math. Soc., **16**, (2001), 323–338.

5.3 Exposition on $\mathscr{D}$-Modules

As in Exposition 5.1 in Chapter V, we will introduce the fundamental notion in the theory of $\mathscr{D}$-modules where $\mathscr{D}$ is the sheaf of differential operators with holomorphic function coefficients over a complex manifold X or $\mathbb{C}^n$. References for Section 5.3 are as follows:

[KashMT] Kashiwara, M., *Algebraic Study of Systems of Partial Differential Equations*, (Master's Thesis, Tokyo University, December 1970), translated by A. D' Agnolo and P. Schneiders, Mémoirs de la Société Mathématique de France, Sér. 2 **63**, (1995), 1–72.

[KashAMS] Kashiwara, M., *D-Modules and Microlocal Calculus*, (Translation of *Daisu Kaiseki Gairon* by Matsumi Saito), Translations of Mathematical Monographs. Vol 217, AMS (2003).

Let $\mathscr{M}$ be a sheaf of germs of $\mathscr{D}$-modules which we call simply a "$\mathscr{D}$-Module". Suppose that the sheaf $\mathscr{M}$ is generated by finitely many $u_1, u_2, \ldots, u_m$ over $\mathscr{D}$. Namely, $\{u_1, u_2, \ldots, u_m\}$ is a set of generators for the $\mathscr{D}$-Module $\mathscr{M}$. Then we have the following epimorphism

$$\mathscr{D}^m \xrightarrow{\cdot u} \mathscr{M} \longrightarrow 0 \tag{3.1}$$

defined by

$$(A_1U_1 \oplus A_2U_2 \oplus \cdots \oplus A_mU_m) \cdot u = A_1u_1 + A_2u_2 + \cdots + A_mu_m, \tag{3.2}$$

where

$$U_j = [0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0], \qquad j = 1, 2, \ldots, m$$

is the canonical basis for the free module $\mathscr{D}^m$. By the Noetherianess of $\mathscr{D}$, $\ker u$ of the epimorphism in (3.1) is also finitely generated over $\mathscr{D}$. Let this epimorphism be $\cdot v$:

$$\mathscr{D}^l \xrightarrow{\cdot v} \ker u \longrightarrow 0 \tag{3.3}$$

where generators $v_1, v_2, \ldots, v_l$ for $\ker u \subset \mathscr{D}^m$ can be written

$$v_j = P_{j1}U_1 + P_{j2}U_2 + \cdots + P_{jm}U_m, \tag{3.4}$$

and the epimorphism $\cdot v$ is given by

$$(B_1V_1 \oplus B_2V_2 \oplus \cdots \oplus B_mV_m) \cdot v = B_1v_1 + B_2v_2 + \cdots + B_mv_m,$$

with

$$V_j = [0, \ldots, 0, \overset{j}{1}, 0, \ldots, 0] \in \mathscr{D}^l, \qquad j = 1, 2, \ldots, m.$$

From (3.1) and (3.3) we obtain

$$\begin{array}{ccccccc} \mathscr{D}^l & & \overset{P}{\dashrightarrow} & & \mathscr{D}^m \xrightarrow{\cdot u} \mathscr{M} \longrightarrow 0 \\ & {\scriptstyle \cdot v}\searrow & & \nearrow{\scriptstyle \iota} & \\ & & \ker u & & \end{array} \tag{3.5}$$

where $P = \iota \circ v$. That is, for $[B_1, B_2, \ldots, B_l] \in \mathscr{D}^l$ we have

$$([B_1, B_2, \ldots, B_l] \cdot P) \cdot u = \big(([B_1, B_2, \ldots, B_l] \cdot v)\iota\big) \cdot u = 0. \tag{3.6}$$

Let P be the $l \times m$-matrix associated with (3.4) with the entries in $\mathscr{D}$. Then for the $m \times 1$-matrix

$$u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}$$

the composition $P \circ u$ of homomorphisms as expressed in (3.6) may be rewritten as follows:

$$\begin{cases} P_{11}u_1 + P_{12}u_2 + \cdots + P_{1m}u_m = 0 \\ P_{21}u_1 + P_{22}u_2 + \cdots + P_{2m}u_m = 0 \\ \quad\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ P_{l\,1}u_1 + P_{l\,2}u_2 + \cdots + P_{l\,m}u_m = 0, \end{cases} \tag{3.7}$$

which is a system of partial differential equations.

Furthermore, beginning at (3.5) we obtain a free resolution of the $\mathscr{D}$-Module $\mathscr{M}$:

$$\begin{array}{ccccccccc} \cdots \longrightarrow \mathscr{D}^r & & \xrightarrow{\quad \cdot Q \quad} & & \mathscr{D}^l & & \xrightarrow{\quad \cdot P \quad} & & \mathscr{D}^m \xrightarrow{\cdot u} \mathscr{M} \longrightarrow 0 \\ & \searrow & & \nearrow & & {\scriptstyle \cdot v}\searrow & & \nearrow{\scriptstyle \iota} & \\ & & \ker P & & & & \ker u & & \\ & \nearrow & & \searrow & & \nearrow & & \searrow & \\ 0 & & & & 0 & & & & 0 \end{array} \tag{3.8}$$

Let $\mathscr{D}\mathscr{M}$ be the category of $\mathscr{D}$-Modules over X where morphisms of $\mathscr{D}\mathscr{M}$ are $\mathscr{D}$-linear homomorphisms. Let $\mathscr{O}$ be the sheaf of germs of holomorphic functions on X. The sheaf $\mathscr{O}$ can be regarded as a $\mathscr{D}$-Module: for $P = \sum_\alpha f_\alpha(z)\partial^\alpha \in \mathscr{D}_z$ (where $f_\alpha(z) \in \mathscr{O}_z$) and $h(z) \in \mathscr{O}_z$ at the stalks at z, $Ph \in \mathscr{O}$ is defined by

$$(Ph)(z) = \sum_\alpha f_\alpha(z)\partial^\alpha h(z),$$

where

$$\partial^\alpha = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial z_n}\right)^{\alpha_n} : \mathscr{O}_z \to \mathscr{O}_z$$

are $\mathbb{C}$-linear partial differential operators. Then we will consider the set

$$\mathrm{Hom}_{\mathscr{D}\mathscr{M}}(\mathscr{M}, \mathscr{O}) = \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}) \tag{3.9}$$

of morphisms in the category of $\mathscr{D}$-Modules. The right hand-side of (3.9) is the sheaf of vector spaces over $\mathbb{C}$ of all $\mathscr{D}$-linear homomorphisms from $\mathscr{M}$ to $\mathscr{O}$. Let $f \in \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ and let $f(u_j) = f_j \in \mathscr{O}$. Then for each $1 \le i \le l$, we have $f(\sum_{j=1}^m P_{ij}u_j) = 0$. Namely,

$$f(\sum P_{ij}u_j) = \sum P_{ij} f(u_j) = \sum P_{ij} f_j = 0.$$

Therefore, morphisms in $\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ may be considered as holomorphic solutions for the system of differential equations expressed as (3.7) of the $\mathscr{D}$-Module $\mathscr{M}$. The left exact contravariant functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})$ is said to be the *solution functor* in $\mathscr{O}$ from the category $\mathscr{D}\mathscr{M}$ of $\mathscr{D}$-Modules. On the other hand, the covariant left exact functor $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \cdot)$ is said to be the *de Rham functor*. The $\mathscr{D}$-Module $\mathscr{O}$ is often said to be the *de Rham Module*. Since we have

$$\mathscr{O} \overset{\approx}{\leftarrow} \mathscr{D} \Big/ \mathscr{D}\left(\frac{\partial}{\partial z_1}\right) + \cdots + \mathscr{D}\left(\frac{\partial}{\partial z_n}\right),$$

we get the free resolution of $\mathscr{O}$:

$$\mathscr{D}^n \xrightarrow{[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}]^{\mathrm{t}}} \mathscr{D} \xrightarrow{\cdot u} \mathscr{O} \longrightarrow 0. \tag{3.10}$$

Namely, as a system of equations we have

$$\begin{cases} \left(\frac{\partial}{\partial z_1}\right)u = 0 \\ \left(\frac{\partial}{\partial z_2}\right)u = 0 \\ \quad \vdots \qquad \vdots \\ \left(\frac{\partial}{\partial z_n}\right)u = 0 \end{cases} \tag{3.11}$$

Such a solution u satisfying (3.11) is a constant. That is, the solution functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})$ takes (3.10) to

$$\begin{array}{ccccccccc} 0 & \to & \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{O}) & \to & \mathscr{H}om_{\mathscr{D}}(\mathscr{D}, \mathscr{O}) & \to & \mathscr{H}om_{\mathscr{D}}(\mathscr{D}^n, \mathscr{O}) & \longrightarrow & \cdots \\ & & \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \\ 0 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathscr{O} & \xrightarrow{[\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}]} & \mathscr{O}^n & \longrightarrow & \cdots . \end{array} \tag{3.12}$$

Namely, the sheaf $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{O})$ of solutions in $\mathscr{O}$ of the de Rham Module $\mathscr{O}$ is the constant sheaf $\mathbb{C}$. In general, for the $\mathscr{D}$-Module $\mathscr{M}$ represented by the free resolution

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\cdot Q} & \mathscr{D}^l & \xrightarrow{\cdot P} & \mathscr{D}^m & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\cdot u} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & \mathscr{M} & \longrightarrow & 0 \end{array}$$

as in (3.8), via the left exact functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})$, we get

$$\begin{array}{ccccccc} 0 \to \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}) & \xrightarrow{\tilde{u}\cdot} & \mathscr{H}om_{\mathscr{D}}(\mathscr{D}^m, \mathscr{O}) & \xrightarrow{\tilde{P}\cdot} & \mathscr{H}om_{\mathscr{D}}(\mathscr{D}^l, \mathscr{O}) & \xrightarrow{\tilde{Q}\cdot} & \cdots \\ & & \downarrow\approx & & \downarrow\approx & & \\ & & \mathscr{O}^m & \longrightarrow & \mathscr{O}^l & \longrightarrow & \cdots . \end{array} \tag{3.13}$$

That is, the $\mathscr{O}$-solution sheaf $\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$ is the sheaf $\ker \tilde{P}$ in (3.13). In terms of notions in Chapters II of derived functors we have

$$\begin{cases} \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}) \approx \mathrm{R}^0\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})\mathscr{M} = \mathscr{H}^0(\mathscr{H}om_{\mathscr{D}}(\mathscr{D}^\bullet, \mathscr{O})) = \ker \tilde{P} \\ \mathscr{E}xt^1_{\mathscr{D}}(\mathscr{M}, \mathscr{O}) \approx \mathrm{R}^1\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})\mathscr{M} = \mathscr{H}^1(\mathscr{H}om_{\mathscr{D}}(\mathscr{D}^\bullet, \mathscr{O})) = \ker \tilde{Q}/\operatorname{im} \tilde{P} \\ \quad\vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \end{cases}$$

where $\mathscr{D}^\bullet$ is any projective (or free) resolution of $\mathscr{M}$. In terms of the notions in Chapter IV on derived categories, for a quasi-isomorphic complex $\mathscr{D}^\bullet$ to a $\mathscr{D}$-Module $\mathscr{M}$, the complex in (3.13) corresponds to

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})\mathscr{M} = \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})$$

so that its j-th cohomology $\mathbb{R}^j\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O}) = \mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M}, \mathscr{O}), j = 0, 1, 2, \ldots$.

In addition to the two references at the beginning of this Exposition, the following books are recommended.

[Bjork] Björk, J.-E., *Analytic $\mathscr{D}$-Modules and Applications*, Kluwer Acad. Publ., 1993.

[Borel] Borel, A., et al, *Algebraic $\mathscr{D}$-Modules*, Perspectives in Math. 2, Academic Press, 1987.

5.4 Cohomological Aspects of $\mathscr{D}$-Modules

The theory of hyperfunctions was developed by Mikio Sato in the 1950's as a generalization of the notion of a Schwartz distribution. See

[Sato] Sato, M., *Theory of Hyperfunctions, I, II*, J. Fac. Sci. Univ. of Tokyo, Sec. I, **8**, (1959), 139–193, 387–437,

Figure 5.3. Sato and the head of the author's son, Kyoto, 1988

where the concept of relative cohomology with the coefficient in the sheaf of holomorphic functions is needed to define the sheaf of hyperfunctions.

We will give a brief discussion on the sheaf $\mathscr{B}$ of (germs of) hyperfunctions and the sheaf $\mathscr{C}$ of (germs of) microfunctions. The serious reader can consult the following book.

[K3] Kashiwara, M., Kawai, T., Kimura, T., *Foundations of Algebraic Analysis*, Princeton Univ. Press, Princeton Math. Series 37, 1986.

Even more ambitious readers can read:

[SKK] Sato, M., Kawai, T., Kashiwara, M., *Microfunctions and Pseudo-Differential Equations*, Lect., Notes in Math., **287**, (1973), Springer-Verlag, 265–529.

Let $\mathscr{O}$ be the sheaf of holomorphic functions over $\mathbb{C}^n$. Then for open sets $W \subset V$ in $\mathbb{C}^n$, we have the restriction homomorphism $\mathscr{O}(V) \to \mathscr{O}(W)$ of abelian groups. As in Subsection 3.4.1, we can interpret this restriction homomorphism as the morphism of global section functors:

$$\Gamma(V, \cdot) \longrightarrow \Gamma(W, \cdot). \tag{4.1}$$

Then define the functor $\Gamma(V, W, \cdot)$ as the kernel of (4.1) evaluated at a sheaf. Namely, we get

$$0 \longrightarrow \Gamma(V, W, \cdot) \longrightarrow \Gamma(V, \cdot) \longrightarrow \Gamma(W, \cdot). \tag{4.2}$$

For a flabby sheaf $\mathscr{F}$ (and for an injective sheaf), by the definition we have the short exact sequence

$$0 \longrightarrow \Gamma(V, W, \mathscr{F}) \longrightarrow \Gamma(V, \mathscr{F}) \longrightarrow \Gamma(W, \mathscr{F}) \longrightarrow 0.$$

(See Subsection 3.4.2 and Notes 17 in Chapter III.) Let $\Omega := \mathbb{R}^n \cap V$. Then by taking $W = V - \Omega$ in (4.2), the exact sequence (4.2) induces the long exact sequence of cohomologies:

$$\begin{aligned} 0 &\longrightarrow \Gamma(V, V-\Omega, \mathscr{O}) \longrightarrow \Gamma(V, \mathscr{O}) \longrightarrow \Gamma(V-\Omega, \mathscr{O}) \longrightarrow \\ &\longrightarrow \mathrm{H}^1(V, V-\Omega, \mathscr{O}) \longrightarrow \mathrm{H}^1(V, \mathscr{O}) \longrightarrow \mathrm{H}^1(V-\Omega, \mathscr{O}) \longrightarrow \\ &\longrightarrow \cdots . \end{aligned} \tag{4.3}$$

Let $V' \subset V$ be another open set in $\mathbb{C}^n$ and let $\Omega' := \mathbb{R}^n \cap V'$. Then we have the restriction homomorphisms

$$\begin{array}{ccccccc} 0 \longrightarrow & \Gamma(V, V-\Omega, \cdot) & \longrightarrow & \Gamma(V, \cdot) & \longrightarrow & \Gamma(V-\Omega, \cdot) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \longrightarrow & \Gamma(V', V'-\Omega', \cdot) & \longrightarrow & \Gamma(V', \cdot) & \longrightarrow & \Gamma(V'-\Omega', \cdot). \end{array}$$

The restriction $\Gamma(V, V-\Omega, \cdot) \to \Gamma(V', V'-\Omega', \cdot)$ induces

$$\begin{array}{ccc} \mathrm{R}^j\Gamma(V, V-\Omega, \mathscr{O}) & \longrightarrow & \mathrm{R}^j\Gamma(V', V'-\Omega', \mathscr{O}) \\ \| & & \| \\ \mathrm{H}^j(V, V-\Omega, \mathscr{O}) & \longrightarrow & \mathrm{H}^j(V', V'-\Omega', \mathscr{O}). \end{array} \tag{4.4}$$

Namely, $V \rightsquigarrow \mathrm{H}^j(V, V-\Omega, \mathscr{O})$ is a presheaf over $\mathbb{C}^n$. Denote the associated sheaf by $\mathscr{H}^j_{\mathbb{R}^n}(\mathscr{O})$. The definition of the sheaf $\mathscr{B}$ of hyperfunctions (of Sato) depends upon the following profound theorems of K. Oka and H. Cartan:

Theorem 19 (Oka's Coherence Theorem)**.** *The sheaf $\mathscr{O}$ is coherent.*

Theorem 20 (Cartan's Theorem)**.** *All the higher cohomologies vanish, i.e.,* $\mathrm{H}^j(V, \mathscr{O}) = 0$*,* $j \geq 1$*, where V is a domain of holomorphy.*

See the following references to understand the meaning of these theorems.

[GrRem] Grauert, H., Remmert, R., *Coherent Analytic Sheaves*. Grundlehren der Mathematischen Wissenschaften 265, Springer-Verlag, 1984.

[Horm] Hörmander, L., *Introduction to Complex Analysis in Several Variables*, North-Holland Math. Library Vol 7, North-Holland Publ. Co., 1973.

[FG] Fritzsche, K., Grauert, H., *From Holomorphic Functions to Complex Manifolds*, Graduate Texts in Mathematics 213, Springer-Verlag, 2002.

Figure 5.4. Kiyoshi Oka. This photo was provided by Mrs. Saori Matsubara (Oka's daughter).

The only non-trivial associated sheaf $\mathscr{H}^n_{\mathbb{R}^n}(\mathscr{O})$ is said to be the sheaf $\mathscr{B}$ of (germs of) Sato's hyperfunctions on $\mathbb{R}^n$, where $\mathscr{H}^j_{\mathbb{R}^n}(\mathscr{O}) = 0$ for $j \neq n$. Note that the hyperfunction sheaf $\mathscr{B} = \mathscr{H}^n_{\mathbb{R}^n}(\mathscr{O})$ is the n-th derived functor of

$$\mathscr{H}^0_{\mathbb{R}^n} : (\text{Sheaves over } \mathbb{C}^n) \rightsquigarrow (\text{Sheaves over } \mathbb{C}^n) :$$

$$\begin{array}{ccc} (\text{Sheaves over } \mathbb{C}^n) & \xrightarrow{\mathscr{H}^0_{\mathbb{R}^n}(\cdot)} & (\text{Sheaves over } \mathbb{C}^n) \\ & \searrow_{\Gamma(V, V-\Omega, \cdot)} & \downarrow{\scriptstyle \Gamma(V, \cdot)} \\ & & \mathsf{Ab}. \end{array} \tag{4.5}$$

Then from (4.5) we get the composite functor spectral sequence

$$E_2^{p,n} = \mathrm{H}^p(V, \mathscr{H}^n_{\mathbb{R}^n}(\mathscr{O}))$$

abutting upon $\mathrm{H}^{p+n}(V, V - \Omega, \mathscr{O})$. In particular, for $p = 0$,

$$E_2^{0,n} = \Gamma(V, \mathscr{H}^n_{\mathbb{R}^n}(\mathscr{O})) \approx \mathrm{H}^n(V, V - \Omega, \mathscr{O})$$

holds, i.e., $V \rightsquigarrow \mathrm{H}^n(V, V - \Omega, \mathscr{O})$ is a sheaf. For an open set U containing V, the excision isomorphism (4.49) in Chapter III implies the isomorphism

$$\mathrm{H}^n(U, U - \Omega, \mathscr{O}) \xrightarrow{\approx} \mathrm{H}^n(V, V - \Omega, \mathscr{O}).$$

In fact, the original idea of M. Sato was to capture a hyperfunction as the sum of boundary values of holomorphic functions. See [K3], [Sato], or

[KaStr] Kato, G., Struppa, D.C., *Fundamentals of Algebraic Microlocal Analysis*, Pure and Applied Math., No. 217, Marcel Dekker Inc, 1999

for details and the historical background. As in (4.2) the sequence of functors

$$0 \longrightarrow \Gamma(V, V - \Omega, \cdot) \longrightarrow \Gamma(V, \cdot) \longrightarrow \Gamma(V - \Omega, \cdot)$$

induces the following triangle corresponding to the long exact sequence (4.3):

$$\begin{array}{ccc} \mathbb{R}\Gamma(V, V - \Omega, \mathscr{O}) & \xleftarrow{\quad [1] \quad} & \mathbb{R}\Gamma(V - \Omega, \mathscr{O}) \\ & \searrow \qquad \nearrow & \\ & \mathbb{R}\Gamma(V, \mathscr{O}) & \end{array}$$

in the derived category $\mathsf{D}(\mathsf{Ab})$. As noted earlier, for $j \neq n$,

$$\mathbb{R}^j \mathscr{H}^0_{\mathbb{R}^n}(\mathscr{O}) = 0 \tag{4.6}$$

and from (4.5) we have $\Gamma(V, V - \Omega, \mathscr{O}) \approx \Gamma(V, \mathscr{H}^0_{\mathbb{R}^n}(\mathscr{O}))$. Therefore from (2.18) in Chapter IV we get

$$\begin{aligned} \mathbb{R}\Gamma(V, V - \Omega, \mathscr{O}) &= \mathbb{R}(\Gamma(V, \cdot) \circ \mathscr{H}^0_{\mathbb{R}^n})\mathscr{O} = \\ &= (\mathbb{R}\Gamma(V, \cdot) \circ \mathbb{R}\mathscr{H}^0_{\mathbb{R}^n})\mathscr{O} = \\ &= \mathbb{R}\Gamma(V, \mathbb{R}\mathscr{H}^0_{\mathbb{R}^n}(\mathscr{O})). \end{aligned}$$

By letting $\mathscr{B}(\Omega) := \mathrm{H}^n(V, V - \Omega, \mathscr{O}) = \Gamma(V, \mathscr{H}^n_{\mathbb{R}^n}(\mathscr{O}))$, the sheaf $\mathscr{B}$ of hyperfunctions can be regarded as a sheaf over $\mathbb{R}^n$. Then $\mathscr{B}$ is a flabby sheaf. Namely, for open sets $\Omega \subset \Omega'$ in $\mathbb{R}^n$, the restriction homomorphism is epimorphic

$$\mathscr{B}(\Omega') \longrightarrow \mathscr{B}(\Omega) \longrightarrow 0.$$

The flabbiness of the hyperfunction sheaf plays an important role in the applications to partial differential equations.

Note that hyperfunctions in one the variable case, i.e., $n = 1$, does not need cohomology. This is because $\mathscr{B}(\Omega) = \mathrm{H}^1(V, V - \Omega, \mathscr{O})$ and $\mathrm{H}^1(V, \mathscr{O}) = 0$ for any V in $\mathbb{C}$. Namely, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \Gamma(V, V - \Omega, \mathscr{O}) \longrightarrow \Gamma(V, \mathscr{O}) \longrightarrow \Gamma(V - \Omega, \mathscr{O}) \longrightarrow \\ \longrightarrow \mathrm{H}^1(V, V - \Omega, \mathscr{O}) \longrightarrow 0 \end{aligned} \tag{4.7}$$

and furthermore, analytic continuation implies $\Gamma(V, \mathscr{O}) \to \Gamma(V - \Omega, \mathscr{O})$ is a monomorphism. That is, by the exactness of (4.7), the global sections of the sheaf of hyperfunctions over $\Omega \subset \mathbb{R}$ become

$$\mathscr{O}(V - \Omega)\big/\mathscr{O}(V) \xrightarrow{\approx} \mathscr{B}(\Omega) = \mathrm{H}^1(V, V - \Omega, \mathscr{O}).$$

For example, if $\Omega = \{0\}$, then $1/z \in \mathscr{O}(V - \{0\})$. The Dirac delta function as hyperfunction is given by the class

$$[1/z] \in \mathscr{O}(V - \{0\})\big/\mathscr{O}(V).$$

See the references mentioned earlier for further topics on hyperfunctions.

Next we will give a definition of the sheaf $\mathscr{C}$ of (germs of) microfunctions defined on the cotangential sphere bundle $\mathsf{S}^*\mathbb{R}^n$. Let

$$\mathsf{S}\mathbb{R}^n \xrightarrow{\ \tau\ } \mathbb{R}^n \xleftarrow{\ \pi\ } \mathsf{S}^*\mathbb{R}^n$$

be the canonical projections onto $\mathbb{R}^n$ from the tangential sphere bundle $\mathsf{S}\mathbb{R}^n$ and the cotangential sphere bundle $\mathsf{S}^*\mathbb{R}^n$. We write $(x, \bar{\eta})$ and $(x, \bar{\xi})$ for points on $\mathsf{S}\mathbb{R}^n$ and $\mathsf{S}^*\mathbb{R}^n$, respectively, where $\bar{\eta} := x + \mathrm{i}\eta 0$ and $\bar{\xi} := x + \mathrm{i}\xi\infty$ as in [K3]. Let $\widetilde{\mathbb{C}^n}$ be a blowing up in $\mathbb{C}^n$ along $\mathbb{R}^n$, i.e., $\widetilde{\mathbb{C}^n}$ can be regarded as the disjoint union $\widetilde{\mathbb{C}^n} = (\mathbb{C}^n - \mathbb{R}^n) \sqcup \mathsf{S}\mathbb{R}^n$. Define

$$\tfrac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n := \{(x, \bar{\xi}, \bar{\eta}) \mid \langle \xi, \eta \rangle \geq 0\}$$

i.e., half of the fibre product of $\mathsf{S}\mathbb{R}^n$ and $\mathsf{S}^*\mathbb{R}^n$. We have

$$\begin{array}{ccccc} & & \tfrac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n & & \\ & \swarrow^{\pi} & & \searrow^{\tau} & \\ \widetilde{\mathbb{C}^n} \hookleftarrow \mathsf{S}\mathbb{R}^n & & & & \mathsf{S}^*\mathbb{R}^n \\ & \searrow^{\tau} & & \swarrow^{\pi} & \\ & & \mathbb{C}^n \hookleftarrow \mathbb{R}^n. & & \end{array}$$

By various purely codimensionality results we can construct the sheaf $\mathscr{C}$ of microfunctions as follows. For the sheaf $\tau^{-1}\mathscr{O}$, $\mathsf{S}\mathbb{R}^n$ is purely 1-codimensional:

$$\mathscr{H}^j_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O}) = 0, \qquad j \neq 1 \tag{4.8}$$

where $\mathscr{H}^j_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})$ is the associated sheaf to the presheaf

$$\tilde{V} \mapsto \mathrm{H}^j(\tilde{V}, \tilde{V} - \mathsf{S}\mathbb{R}^n \cap \tilde{V}, \tau^{-1}\mathscr{O})$$

for an open set $\tilde{V}$ in $\widetilde{\mathbb{C}^n}$. (This is Proposition 2.1.1 of [K3].) Next, for the sheaf $\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})$ over the above $\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n$, the projection $\tau : \frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n \to \mathsf{S}^*\mathbb{R}^n$ is purely $(n-1)$-codimensional in the following sense:

$$\mathrm{R}^j\tau_*(\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})) = 0, \qquad j \neq n-1. \tag{4.9}$$

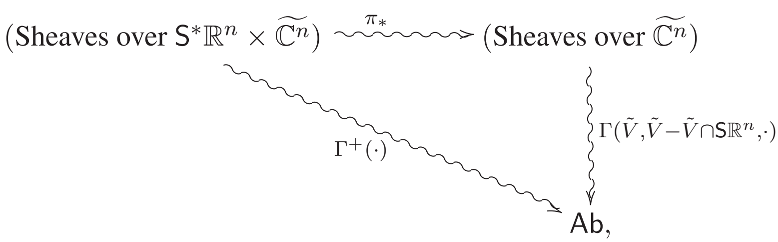
(This is Proposition 2.1.2′ in [K3].) Then the sheaf $\mathscr{C}$ is defined by

$$\mathscr{C} := \mathrm{R}^{n-1}\tau_*(\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})). \tag{4.10}$$

Next we will prove that the sheaf $\mathscr{C}$ as defined in (4.10) can also be expressed as $\mathscr{H}^n_{\mathsf{S}^*\mathbb{R}^n}(\pi^{-1}\mathscr{O})$. For the projection

$$\mathsf{S}^*\mathbb{R}^n \times \widetilde{\mathbb{C}^n} \longrightarrow \widetilde{\mathbb{C}^n},$$

apply the Leray spectral sequence in Subsection 3.4.7 to



where $\Gamma^+(\cdot) := \Gamma(\tilde{V} \times V^*, \tilde{V} \times V^* - \tilde{V} \times V^* \cap \mathsf{S}\mathbb{R}^n, \cdot)$. That is, the initial term is given as

$$E_2^{p,q} = \mathrm{H}^p(\tilde{V}, \tilde{V} - \tilde{V} \cap \mathsf{S}\mathbb{R}^n, \mathrm{R}^q\pi_*(\pi^{-1}(\tau^{-1}\mathscr{O})))$$

abutting upon

$$\mathrm{H}^n(\tilde{V} \times V^*, \tilde{V} \times V^* - \tilde{V} \times V^* \cap \mathsf{S}\mathbb{R}^n, \pi^{-1}(\tau^{-1}\mathscr{O}))$$

where $\tilde{V}$ and V^* are open sets in $\widetilde{\mathbb{C}^n}$ and $\mathsf{S}^*\mathbb{R}^n$, respectively. we can take contractible V^* so that $\mathrm{H}^p(V^*, \tau^{-1}\mathscr{O}) = 0$ for $p \neq 0$. Then we get

$$\mathrm{R}^q\pi_*(\pi^{-1}\tau^{-1}\mathscr{O}) = \begin{cases} \tau^{-1}\mathscr{O} & \text{for } q = 0 \\ 0 & \text{for } q \neq 0. \end{cases}$$

Namely, only $E_2^{p,0}$ are non-trivial:

$$E_2^{p,0} = \mathrm{H}^p(\tilde{V}, \tilde{V} - \tilde{V} \cap \mathsf{S}\mathbb{R}^n, \tau^{-1}\mathscr{O}).$$

From the purely 1-codimensionality of $\mathsf{S}\mathbb{R}^n$ for $\tau^{-1}\mathscr{O}$, i.e., (4.8), we get

$$\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O}) \xrightarrow{\approx} \mathscr{H}^1_{\pi^{-1}(\mathsf{S}\mathbb{R}^n)}(\pi^{-1}(\tau^{-1}\mathscr{O})) \approx \mathscr{H}^1_{\frac{1}{2}\mathsf{S}^*\mathbb{R}^n}(\pi^{-1}(\tau^{-1}\mathscr{O})).$$

Let us compute the higher direct image

$$E_2^{p,1} = \mathrm{R}^p\tau_*(\mathscr{H}^1_{\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n}(\pi^{-1}(\tau^{-1}\mathscr{O})).$$

Since we have

$$0 = E_2^{p-2,2} \longrightarrow E_2^{p,1} \longrightarrow E_2^{p+2,0} = 0,$$

$E_2^{p,1} \xrightarrow{\approx} E_\infty^{p,1}$ is isomorphic to the abutment

$$\mathrm{R}^{p+1}(\tau_*\mathscr{H}^0_{\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n})(\pi^{-1}(\tau^{-1}\mathscr{O})).$$

Again by the pure $(n-1)$-codimensionality of $\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})$ for τ, i.e., (4.9), we get

$$E_2^{n-1,1} = \mathrm{R}^{n-1}\tau_*(\mathscr{H}^1_{\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n}(\pi^{-1}(\tau^{-1}\mathscr{O}))).$$

Then the abutment becomes

$$\mathrm{R}^n(\tau_*\mathscr{H}^0_{\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n})(\pi^{-1}(\tau^{-1}\mathscr{O})).$$

Since we have $\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n = \tau^{-1}(\mathsf{S}^*\mathbb{R}^n)$ we get

$$\mathrm{R}^n(\tau_*\mathscr{H}^0_{\frac{1}{2}\mathsf{S}^*\mathsf{S}\mathbb{R}^n})(\pi^{-1}(\tau^{-1}\mathscr{O})) \approx \mathrm{R}^n(\mathscr{H}^0_{\mathsf{S}^*\mathbb{R}^n}\tau_*)(\tau^{-1}(\pi^{-1}\mathscr{O})).$$

Namely, we get the sheaf

$$\mathscr{C} = \mathrm{R}^{n-1}\tau_*(\pi^{-1}\mathscr{H}^1_{\mathsf{S}\mathbb{R}^n}(\tau^{-1}\mathscr{O})) \approx \mathrm{R}^n\mathscr{H}^0_{\mathsf{S}^*\mathbb{R}^n}(\pi^{-1}\mathscr{O}),$$

i.e., we obtain

$$\mathscr{C} \approx \mathscr{H}^n_{\mathsf{S}^*\mathbb{R}^n}(\pi^{-1}\mathscr{O}).$$

See any reference mentioned above for the fundamental exact sequence of sheaves $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ of real analytic functions, hyperfunctions and microfunctions:

$$0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_*\mathscr{C} \longrightarrow 0.$$

5.4.1 The de Rham Functor

Let $\mathscr{D}$ be the sheaf of germs of holomorphic (linear) differential operators over an n-dimensional complex manifold X. Then we have two functors: the solution functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O})$, a contravariant functor from the category $\mathscr{DM}$ of $\mathscr{D}$-Modules to the category $\mathbb{C}\mathscr{V}$ of sheaves of $\mathbb{C}$-vector spaces. Namely, we have

$$\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{O}) : \mathscr{DM} \rightsquigarrow \mathbb{C}\mathscr{V}$$

such that at $x \in X$ the stalk $\mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{O})_x = \mathrm{Hom}_{\mathscr{D}_x}(\mathscr{M}_x, \mathscr{O}_x)$ is a module over $\mathbb{C}_x$ of the constant sheaf $\mathbb{C}$. The other functor $\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \cdot)$ is covariant from $\mathscr{DM}$ to $\mathbb{C}\mathscr{V}$ which has also been mentioned in Exposition 5.3 and called the *de Rham functor*. Then define the complex of sheaves of p-forms with coefficients in a $\mathscr{D}$-Module $\mathscr{M}$ as follows:

$$\Omega^\bullet(\mathscr{M}) := \Omega^\bullet \otimes_{\mathscr{O}} \mathscr{M} \approx \mathrm{Hom}_{\mathscr{O}}(\wedge^\bullet \Theta, \mathscr{M}),$$

where

$$\mathrm{d}^p_{\mathscr{M}} : \Omega^p \otimes_{\mathscr{O}} \mathscr{M} \longrightarrow \Omega^{p+1} \otimes_{\mathscr{O}} \mathscr{M}$$

is defined by

$$\mathrm{d}^p_{\mathscr{M}}(\omega \otimes m) = \mathrm{d}^p\omega \otimes m + \sum_{i=1}^{n} (\mathrm{d}^p x_i \wedge \omega) \otimes \big(\tfrac{\partial}{\partial x_i}\big) m.$$

Note that $\Theta = \mathscr{D}^{(1)}$ is the holomorphic tangent sheaf, i.e., $f \in \mathscr{D}^{(1)}_x$ can be written as

$$f_1\big(\tfrac{\partial}{\partial x_1}\big) + f_2\big(\tfrac{\partial}{\partial x_2}\big) + \cdots + f_n\big(\tfrac{\partial}{\partial x_n}\big)$$

using local coordinates $(x_1, x_2, \ldots, x_n)$. Also, $\Omega^\bullet$ is the sheaf of holomorphic p-forms on X. Then by replacing $\mathscr{M}$ by $\mathscr{D}$ we get the following free resolution (4.11) of the right $\mathscr{D}$-Module Ω^n:

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathscr{D} & \xrightarrow{\mathrm{d}^0_{\mathscr{D}}} & \Omega^1 \otimes_{\mathscr{O}} \mathscr{D} & \xrightarrow{\mathrm{d}^1_{\mathscr{D}}} & \cdots & \longrightarrow & \Omega^n \otimes_{\mathscr{O}} \mathscr{D} & \longrightarrow & 0 \\ & & & & & & \downarrow \epsilon & & \\ & & & & & & \Omega^n & & \end{array} \tag{4.11}$$

where

$$\begin{cases} \mathrm{d}^0_{\mathscr{D}}(1_{\mathscr{D}}) & = \sum_{i=1}^n \mathrm{d}x_i \otimes \big(\tfrac{\partial}{\partial x_i}\big), \text{ and} \\ \mathrm{d}^1_{\mathscr{D}}(\omega \otimes 1_{\mathscr{D}}) & = \mathrm{d}^1(\omega) \otimes 1_{\mathscr{D}} - \omega \otimes \mathrm{d}^0_{\mathscr{D}}(1_{\mathscr{D}}) \end{cases} \tag{4.12}$$

and in general

$$\mathrm{d}^p_{\mathscr{D}}(\omega \otimes 1_{\mathscr{D}}) = \mathrm{d}^p(\omega) \otimes 1_{\mathscr{D}} + (-1)^p \omega \otimes \mathrm{d}^0_{\mathscr{D}}(1_{\mathscr{D}}) \tag{4.13}$$

for $\omega \in \Omega^p$ and $1_{\mathscr{D}} \in \mathscr{D}$. A right $\mathscr{D}$-Module structure of Ω^n, the highest form on X, is defined by

$$\big(f \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n\big)\big(\tfrac{\partial}{\partial x_i}\big) = \big(\tfrac{\partial}{\partial x_i}\big)(f)\mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n. \tag{4.14}$$

Note also that the augmentation $\epsilon : \Omega^n \otimes \mathscr{D} \to \Omega^n$ in (4.11) is defined by the right $\mathscr{D}$-Module structure of Ω^n in (4.14). On the other hand, $\Theta = \mathscr{D}^{(1)}$ is a free $\mathscr{O}$-Module as we noted. The Koszul complex $\wedge^\bullet(\mathscr{D}^n)$ associated with $(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$ becomes a free (projective) resolution of $\mathscr{O}$:

$$\begin{array}{l} 0 \leftarrow \mathscr{D} \xleftarrow{\delta^1} \mathscr{D} \otimes_{\mathscr{O}} \wedge^1\Theta \xleftarrow{\delta^2} \mathscr{D} \otimes_{\mathscr{O}} \wedge^2\Theta \leftarrow \cdots \leftarrow \mathscr{D} \otimes_{\mathscr{O}} \wedge^n\Theta \leftarrow 0 \\ \big\downarrow \epsilon' \\ \mathscr{O} \end{array} \tag{4.15}$$

where the augmentation ϵ' is defined by $\epsilon'(1_{\mathscr{D}}) = 1_{\mathscr{D}} \in \mathscr{O}$, i.e., $\epsilon(P \otimes u) = Pu$ for $P \in \mathscr{D}$ and $u \in \mathscr{O}$. The morphism $\delta^q : \mathscr{D} \otimes_{\mathscr{O}} \wedge^q\Theta \to \mathscr{D} \otimes_{\mathscr{O}} \wedge^{q-1}\Theta$ is defined by

$$\begin{aligned} \delta^q(P \otimes (\theta_1 \wedge \cdots \wedge \theta_q)) &= \sum_{i=1}^{q} (-1)^{i-1} P\theta_i \otimes (\theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge_q) \\ &+ \sum_{1 \le i < k \le q} (-1)^{i+k} P \otimes ([\theta_i, \theta_k] \wedge \theta_1 \wedge \cdots \wedge \hat{\theta}_i \wedge \cdots \wedge \hat{\theta}_k \wedge \cdots \wedge \theta_q) \end{aligned}$$

for $P \in \mathscr{D}$ and $\theta_i \in \Theta$, $i = 1, 2, \dots, n$, where $[\theta_i, \theta_k] = \theta_i\theta_k - \theta_k\theta_i$. In particular, $\delta^1(P \otimes \theta) = P\theta$. Namely, $\operatorname{im} \delta^1 = \sum_{i=1}^n \mathscr{D}\big(\frac{\partial}{\partial x_i}\big)$. Consequently, the 0-th homology of (4.15) is the isomorphism induced by the augmentation ϵ'. That is, $\mathscr{O} \xleftarrow{\approx} \mathscr{D}/\big(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\big)$.

Let us extend $\mathscr{H}om_{\mathscr{D}}(\cdot, \cdot)$ to functors from the category $\mathscr{DM}$ of left $\mathscr{D}$-Modules to the derived category $\mathsf{D}(\mathscr{DM})$. Namely, we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\cdot, \cdot) : \mathsf{D}(\mathscr{DM}) \rightsquigarrow \mathsf{D}(\mathbb{C}\mathscr{V}).$$

For example, $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{M})$ can be computed by the free resolution (4.15) of $\mathscr{O}$ as

$$\begin{aligned} \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{M}) &= \mathscr{H}om_{\mathscr{D}}(\mathscr{D} \otimes_{\mathscr{O}} \wedge^\bullet\Theta, \mathscr{M}) \approx \\ &\approx \mathscr{H}om_{\mathscr{O}}(\wedge^\bullet\Theta, \mathscr{M}) \approx \\ &\approx \Omega^\bullet \otimes_{\mathscr{O}} \mathscr{M} = \Omega^\bullet(\mathscr{M}). \end{aligned}$$

Therefore, in terms of the cohomologies we have

$$\mathscr{E}xt^i_{\mathscr{D}}(\mathscr{O}, \mathscr{M}) = \mathscr{H}^j(\Omega^\bullet(\mathscr{M})).$$

Note also that the contravariant functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{D})$ gives the free resolution of Ω^n as in (4.11):

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{D}) \approx \Omega^\bullet \otimes_{\mathscr{O}} \mathscr{D}.$$

Namely, we get

$$\begin{cases} \mathbb{R}^j \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{D}) = \mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}, \mathscr{D}) = 0 & \text{for } j \neq n \\ \mathbb{R}^n \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{D}) = \mathscr{E}xt^n_{\mathscr{D}}(\mathscr{O}, \mathscr{D}) \approx \Omega^n. \end{cases} \tag{4.16}$$

Conversely, (4.11) can be used to compute $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\Omega^n, \mathscr{D})$. Namely we obtain the free resolution (4.15) of $\mathscr{O}$ by the functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{D})$ via (4.11):

$$\begin{cases} \mathbb{R}^j \mathscr{H}om_{\mathscr{D}}(\Omega^n, \mathscr{D}) = \mathscr{E}xt^j_{\mathscr{D}}(\Omega^n, \mathscr{D}) = 0 & \text{for } j \neq n \\ \mathbb{R}^n \mathscr{H}om_{\mathscr{D}}(\Omega^n, \mathscr{D}) \approx \mathscr{O}. \end{cases} \tag{4.17}$$

Notice that from (4.16) and (4.17) we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{D}), \mathscr{D}) \approx \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\Omega^n, \mathscr{D}) \approx \mathscr{O}, \tag{4.18}$$

i.e., $\mathscr{E}xt^n_{\mathscr{D}}(\mathscr{E}xt^n_{\mathscr{D}}(\mathscr{O}, \mathscr{D}), \mathscr{D}) \approx \mathscr{O}$. Also notice that for the free resolution $\Omega^\bullet(\mathscr{D}) \xrightarrow{\epsilon} \Omega^n$ of Ω^n as given in (4.11), the *right* exact functor $\cdot \otimes_{\mathscr{D}} \mathscr{M}$ induces

$$\begin{array}{ccccccccc} 0 \to \Omega^0(\mathscr{D}) \otimes \mathscr{M} & \to & \Omega^1(\mathscr{D}) \otimes \mathscr{M} & \to \cdots \to & \Omega^n(\mathscr{D}) \otimes \mathscr{M} & \xrightarrow{\tilde{\epsilon}} & \Omega^n \otimes \mathscr{M} \\ \downarrow \approx & & \downarrow \approx & & \downarrow \approx & & \| \\ 0 \longrightarrow \Omega^0 \otimes \mathscr{M} & \longrightarrow & \Omega^1 \otimes \mathscr{M} & \longrightarrow \cdots \longrightarrow & \Omega^n \otimes \mathscr{M} & \longrightarrow & \Omega^n \otimes \mathscr{M}, \end{array} \tag{4.19}$$

where in the upper sequence tensor products are over $\mathscr{D}$ and in the lower, over $\mathscr{O}$. The homology and the cohomology of the sequence in (4.19) provide the isomorphism

$$\mathscr{T}or^{\mathscr{D}}_{n-j}(\Omega^n, \mathscr{M}) \xrightarrow{\approx} \mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}, \mathscr{M}), \tag{4.20}$$

or $\Omega^n \overset{\mathbb{L}}{\otimes}_{\mathscr{D}} \mathscr{M} \approx \mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{M})$ in terms of derived category notion. Furthermore, for $\mathscr{M} = \mathscr{O}$ we have

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{O}) \approx \Omega^\bullet.$$

By the Poincaré Lemma we have

$$\mathbb{R}^j \mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{O}) = \mathscr{E}xt^j_{\mathscr{D}}(\mathscr{O}, \mathscr{O}) = \mathscr{H}^j(\Omega^\bullet) = \begin{cases} 0, & j \neq 0 \\ \mathbb{C}, & j = 0. \end{cases}$$

That is, as an object of $\mathsf{D}(\mathbb{C}\mathscr{V})$, $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O},\mathscr{O})$ is isomorphic to $\mathbb{C}$.

Let $\mathscr{N}$ be a left $\mathscr{D}$-Module and let $\mathcal{P}_\bullet \to \mathscr{N}$ be a projective resolution of $\mathscr{N}$. Then we have

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{N},\mathscr{D}) = \mathscr{H}^j(\mathscr{H}om_{\mathscr{D}}(\mathcal{P}_\bullet,\mathscr{D})).$$

Let $'\mathcal{P}_\bullet$ be a complex of flat right $\mathscr{D}$-Modules quasi-isomorphic to the complex $\mathscr{H}om_{\mathscr{D}}(\mathcal{P}_\bullet,\mathscr{D})$, i.e., $\mathscr{H}^j('\mathcal{P}_\bullet) \approx \mathscr{H}^j(\mathscr{H}om_{\mathscr{D}}(\mathcal{P}_\bullet,\mathscr{D}))$. Then let $\mathscr{C}^j(\mathscr{N},\mathscr{M})$ be the functor, contravariant in $\mathscr{N}$ and covariant in $\mathscr{M}$ defined by

$$\mathscr{H}^j('\mathcal{P}_\bullet \otimes_{\mathscr{D}} \mathscr{M}).$$

Then $\mathscr{C}^j(\mathscr{N},\mathscr{M})$ is an exact connected sequence of functors. We have the following spectral sequence abutting upon $\mathscr{C}^j(\mathscr{N},\mathscr{M})$:

$$E_2^{p,q} = \mathscr{T}or^{\mathscr{D}}_{-p}(\mathscr{E}xt^q_{\mathscr{D}}(\mathscr{N},\mathscr{D}),\mathscr{M}). \tag{4.21}$$

Note that for a finitely presented $\mathscr{D}$-Module $\mathscr{M}$ (i.e., $\mathscr{M}$ is coherent as a $\mathscr{D}$-Module), $\mathscr{C}^j(\mathscr{N},\mathscr{M})$ becomes $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{N},\mathscr{M})$. This is because we have

$$\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D}) \otimes_{\mathscr{D}} \mathscr{M} = \mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{M})$$

for a finitely generated projective $\mathscr{D}$-Module $\mathscr{M}$. As an object of the derived category, it is

$$\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{N},\mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}} \mathscr{M}$$

inducing the spectral sequence (4.21). For $\mathscr{N} = \mathscr{O}$ and a coherent $\mathscr{D}$-Module $\mathscr{M}$, the spectral sequence (4.21) gives the isomorphism in (4.20). Notice also that the universal coefficient spectral sequence (2.38) and the spectral sequence (4.21) associated to $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{N},\mathscr{D}) \overset{\mathbb{L}}{\otimes}_{\mathscr{D}} \mathscr{M}$ are essentially the same. See [KaStr] for details.

5.4.2 Cohomological Characterization of Holonomic $\mathscr{D}$-Modules

The notion of the characteristic variety $V(\mathscr{M})$ of a $\mathscr{D}$-Module $\mathscr{M}$ is central for the microlocal analysis of $\mathscr{M}$. The holonomicity of $\mathscr{M}$ is defined in terms of the dimension of the characteristic variety $V(\mathscr{M})$. A $\mathscr{D}$-Module $\mathscr{M}$ is said to be *holonomic* if the dimension of $V(\mathscr{M})$ is the smallest possible, i.e., $\dim V(\mathscr{M}) = n$. Such a system of partial differential equations is called a maximally overdetermined system. It is known that the holonomicity condition is equivalent to

$$\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M},\mathscr{D}) = 0, \qquad j \neq n.$$

(See any of the references for the proof.) We have already observed in (4.18) that $\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathbb{R}\mathscr{H}om_{\mathscr{D}}(\mathscr{O}, \mathscr{D}), \mathscr{D}) = \mathscr{O}$. We will prove that for a holonomic $\mathscr{D}$-Module $\mathscr{M}$ we have

$$\mathscr{E}xt^n_{\mathscr{D}}(\mathscr{E}xt^n_{\mathscr{D}}(\mathscr{M}, \mathscr{D}), \mathscr{D}) \approx \mathscr{M}. \tag{4.22}$$

Note that $\mathscr{E}xt^j_{\mathscr{D}}(\mathscr{M}, \mathscr{D}) = 0$ for $j \neq n$ implies the following: for an exact sequence of holonomic left $\mathscr{D}$-Modules

$$0 \longrightarrow \mathscr{M}' \longrightarrow \mathscr{M} \longrightarrow \mathscr{M}'' \longrightarrow 0,$$

we get

$$0 \longrightarrow \mathscr{E}xt^n_{\mathscr{D}}(\mathscr{M}'', \mathscr{D}) \longrightarrow \mathscr{E}xt^n_{\mathscr{D}}(\mathscr{M}, \mathscr{D}) \longrightarrow \mathscr{E}xt^n_{\mathscr{D}}(\mathscr{M}', \mathscr{D}) \longrightarrow 0.$$

That is, the contravariant functor $\mathscr{E}xt^n_{\mathscr{D}}(\cdot, \mathscr{D})$ from the category of left holonomic $\mathscr{D}$-Modules to the category of right holonomic $\mathscr{D}$-Modules is an exact functor.

In order to prove (4.22), first take a projective resolution of $\mathscr{M}$ as

$$\mathcal{P}_\bullet \xrightarrow{\epsilon} \mathscr{M}. \tag{4.23}$$

By the contravariant functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{D})$, from (4.23) we get the complex

$$\begin{array}{c} {}'\mathcal{P}^\bullet := \mathscr{H}om_{\mathscr{D}}(\mathcal{P}_\bullet, \mathscr{D}) \\ \Big\uparrow {\scriptstyle \mathscr{H}om_{\mathscr{D}}(\epsilon, \mathscr{D})} \\ \mathscr{H}om_{\mathscr{D}}(\mathscr{M}, \mathscr{D}). \end{array}$$

Then let

$$\begin{array}{ccccccccc}
{}'\mathcal{P}^0 & \longrightarrow & {}'\mathcal{P}^1 & \longrightarrow & \cdots & \longrightarrow & {}'\mathcal{P}^j & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{Q}^{0,0} & \longrightarrow & \mathcal{Q}^{1,0} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}^{j,0} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \\
\mathcal{Q}^{0,-1} & \longrightarrow & \mathcal{Q}^{1,-1} & \longrightarrow & \cdots & \longrightarrow & \mathcal{Q}^{j,-1} & \longrightarrow & \cdots \\
\uparrow & & \uparrow & & & & \uparrow & & \\
\vdots & & \vdots & & & & \vdots & &
\end{array} \tag{4.24}$$

be Cartan–Eilenberg resolution of ${}'\mathcal{P}^\bullet$ with projective object $\mathcal{Q}^{\bullet,\bullet}$. Then the contravariant functor $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{D})$ carries the double complex (4.24) in the fourth quadrant to the following double complex in the second quadrant:

$$\begin{array}{ccccccccc} & & \vdots & & & & \vdots & & \vdots \\ & & \uparrow & & & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & {}'\mathcal{Q}^{-j,1} & \longrightarrow & \cdots & \longrightarrow & {}'\mathcal{Q}^{-1,1} & \longrightarrow & {}'\mathcal{Q}^{0,1} \\ & & \uparrow & & & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & {}'\mathcal{Q}^{-j,0} & \longrightarrow & \cdots & \longrightarrow & {}'\mathcal{Q}^{-1,0} & \longrightarrow & {}'\mathcal{Q}^{0,0} \\ & & \uparrow & & & & \uparrow & & \uparrow \\ \cdots & \longrightarrow & {}''\mathcal{P}^{-j,0} & \longrightarrow & \cdots & \longrightarrow & {}''\mathcal{P}^{-1,0} & \longrightarrow & {}''\mathcal{P}^{0,0} \end{array} \tag{4.25}$$

where ${}''\mathcal{P}^{-j,0} := \mathscr{H}om_{\mathscr{D}}({}'\mathcal{P}^{j}, \mathscr{D})$ and ${}'\mathcal{Q}^{-j,i} := \mathscr{H}om_{\mathscr{D}}(\mathcal{Q}^{j,-i}, \mathscr{D})$. Then we have the spectral sequences associated with the double complex (4.25). Namely, by (3.7), (3.8), (3.9),

$$\begin{cases} E_0^{-p,q} = {}'\mathcal{Q}^{-p,q}, & {}'E_0^{p,-q} = {}'\mathcal{Q}^{-q,p} \\ E_1^{-p,q} = \mathscr{H}_{\uparrow}^{q}({}'\mathcal{Q}^{-p,\bullet}), & {}'E_1^{p,-q} = \mathscr{H}_{\rightarrow}^{-q}({}'\mathcal{Q}^{\bullet,p}) \\ E_2^{-p,q} = \mathscr{H}_{\rightarrow}^{-p}(\mathscr{H}_{\uparrow}^{q}({}'\mathcal{Q}^{\bullet,\bullet})), & {}'E_2^{p,-q} = \mathscr{H}_{\uparrow}^{p}(\mathscr{H}_{\rightarrow}^{-q}({}'\mathcal{Q}^{\bullet,\bullet})) \end{cases} \tag{4.26}$$

both abutting upon the total cohomology $\mathscr{H}^n({}'\mathcal{Q}^{\bullet})$ where

$${}'\mathcal{Q}^n := \bigoplus_{p+q=n} {}'\mathcal{Q}^{-p,q}.$$

Since $\mathscr{H}om_{\mathscr{D}}(\cdot, \mathscr{D})$ is a left exact functor we have

$$E_1^{-p,0} = \mathscr{H}_{\uparrow}^{0}({}'\mathcal{Q}^{-p,\bullet}) \approx \mathscr{H}om_{\mathscr{D}}({}'\mathcal{P}^{p,0}, \mathscr{D}).$$

Moreover, for the projective object ${}'\mathcal{P}^{p,0}$, we get

$${}''\mathcal{P}^{-p,0} = \mathscr{H}om_{\mathscr{D}}({}'\mathcal{P}^{p,0}, \mathscr{D}) = \mathscr{H}om_{\mathscr{D}}(\mathscr{H}om_{\mathscr{D}}(\mathcal{P}^{-p}, \mathscr{D}), \mathscr{D}) \approx \mathcal{P}^{-p}.$$

Then from

$$\begin{array}{ccccccc} \cdots & \longrightarrow & E_1^{-p,0} & \longrightarrow & E_1^{-(p-1),0} & \longrightarrow & \cdots \\ & & \| & & \| & & \\ \cdots & \longrightarrow & \mathcal{P}^{-p} & \longrightarrow & \mathcal{P}^{-(p-1)} & \longrightarrow & \cdots, \end{array}$$

we get $E_2^{-p,0} = 0$ for $p \neq 0$, and we have $E_2^{0,0} \approx \mathscr{M}$. Note that the diagram (4.25) is vertically exact; i.e., for $q \neq 0$ we have $E_1^{-p,q} = \mathscr{H}_{\uparrow}^{q}({}'\mathcal{Q}^{-p,\bullet}) = 0$.

Let us re-write $\mathscr{H}^q_\uparrow({}'\mathcal{Q}^{-p,\bullet})$ as follows:

$$\begin{aligned}\mathscr{H}^q_\uparrow({}'\mathcal{Q}^{-p,\bullet}) &= \mathrm{R}^q\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D}){}'\mathcal{P}^p = \\ &= \mathrm{R}^q\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})(\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})\mathcal{P}^{-p}).\end{aligned}$$

That is, $E_1^{-p,q}$ associated with the double complex (4.25) can be considered as the $E_2^{q,0}$-term of the composite functor of

$$\begin{array}{ccc} \text{left}-\mathscr{DM} & \xrightarrow{\ \mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})\ } & \text{right}-\mathscr{DM} \\ & \searrow & \Big\downarrow{\scriptstyle \mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})} \\ & & \text{left}-\mathscr{DM}. \end{array} \tag{4.27}$$

Namely, we have

$$E_1^{-p,q}(4.25) = E_2^{q,0} = \mathrm{R}^q\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})(\mathrm{R}^0\mathscr{H}om_{\mathscr{D}}(\cdot,\mathscr{D})(\mathcal{P}^{-p})).$$

However, for the projective object $\mathcal{P}^{-p}$, we have

$$\mathscr{H}om_{\mathscr{D}}(\mathscr{H}om_{\mathscr{D}}(\mathcal{P}^{-p},\mathscr{D}),\mathscr{D}) \approx \mathcal{P}^{-p},$$

i.e., the composition of two functors in (4.27) is an identity functor on projectives. Since an identity is an exact functor, the abutment $E^q = 0$ for $q \geq 1$. Therefore, $E_2^{q,0} = 0$ for $q \geq 1$. Consequently, $E_1^{-p,q} = \mathscr{H}^q_\uparrow({}'\mathcal{Q}^{-p,\bullet}) = 0$ for $q \geq 1$. Then

$$0 = E_2^{-2,1}(4.25) \longrightarrow E_2^{0,0}(4.25) \longrightarrow E_2^{2,-1}(4.25) = 0$$

implies $\mathscr{M} \approx E_\infty^{0,0} \approx E^0 = \mathscr{H}^0({}'\mathcal{Q}^\bullet)$ of (4.25).

Let us compute the second spectral sequence in (4.26) induced by the filtration as in (3.23). We will draw diagrams as we did in Note 16 in Chapter III. At the level zero, $\{{}'E_0^{p,-q}\}$ can be shown as

$$\begin{array}{ccccccc} \cdots & \longrightarrow & {}'E_0^{p,-q} & \longrightarrow & {}'E_0^{p,-q-1} & \longrightarrow & \cdots \\ & & \| & & \| & & \\ \cdots & \longrightarrow & {}'\mathcal{Q}^{-q,p} & \longrightarrow & {}'\mathcal{Q}^{-q-1,p} & \longrightarrow & \cdots\ . \end{array}$$

Then we have

$$\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
{}'E_1^{2,-q} & = & \mathscr{H}_{\rightarrow}^{-q}({}'\mathcal{Q}^{\bullet,2}) \\
\uparrow & & \uparrow \\
{}'E_1^{1,-q} & = & \mathscr{H}_{\rightarrow}^{-q}({}'\mathcal{Q}^{\bullet,1}) \\
\uparrow & & \uparrow \\
{}'E_1^{0,-q} & = & \mathscr{H}_{\rightarrow}^{-q}({}'\mathcal{Q}^{\bullet,0}) \\
\uparrow & & \uparrow \\
0 & & 0
\end{array}$$

and

$${}'E_2^{p,q} = \mathscr{E}xt^p_{\mathscr{D}}(\mathscr{E}xt^q_{\mathscr{D}}(\mathscr{M}, \mathscr{D}), \mathscr{D})$$

abutting upon that $(p-q)$-th derived functor E^{p-q} of the identity, i.e., $E^0 \approx \mathscr{M}$ for $p-q=0$. From the sequence

$$\begin{array}{ccccccc}
\cdots \longrightarrow & {}'E_2^{p-2,-p+1} & \longrightarrow & {}'E_2^{p,-q} & \longrightarrow & {}'E_2^{p+2,-q-1} & \longrightarrow \cdots \\
 & \| & & & & \| & \\
 & 0 & & & & 0 &
\end{array}$$

and from ${}'E_\infty^{p,-p} = \mathsf{F}^p(E^0)/\mathsf{F}^{p+1}(E^0) = \mathsf{F}^p(\mathscr{M})/\mathsf{F}^{p+1}(\mathscr{M})$ where $\mathscr{M} = E^0 = \bigoplus_{p=0} E_\infty^{p,-p}$, we get

$${}'E_2^{n,n} = \mathscr{E}xt^n_{\mathscr{D}}(\mathscr{E}xt^n_{\mathscr{D}}(\mathscr{M}, \mathscr{D}), \mathscr{D}) \approx E_\infty^{n,-n} \approx \mathscr{M}$$

for a holonomic $\mathscr{D}$-Module $\mathscr{M}$.

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EPILOGUE (INFORMAL)

Thank you, Fred, for inviting me to Antwerpen and for suggesting such a charming title as *The Heart of Cohomology*.

To Daniel I send “Thanks" for working with me even while so much was going on in your life.

Thank you, Marieke Mol for teaching me the innocent poem by Paul van Ostaijen.

Thank you, Chrissie, for checking my English, and Alex for playing great piano music of Bach, Beethoven, Mozart, . . . , everyday.

Please allow me to sing a beautiful poem from the Manyoushu Vol 1, 20:

Akanesasu Murasaki Noyuki Shimenoyuki
Numoriwa Mizuya Kimigasodehuru

To the young reader: only elemental $5 - 7 - 5 - 7 - 7$ syllables like the above Manyoushu poem sometimes can sing the beauty.

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