On the Dirac-Maxwell correspondence

Preprint · April 2021

DOI: 10.13140/RG.2.2.30297.03688

A. I. Arbab
Qassim University

On the Dirac-Maxwell correspondence

A. I. Arbab [a]

Department of Physics, College of Science, Qassim University, P.O. Box 6644, Buraidah 51452, Saudi Arabia

Department of Physics, Faculty of Science, University of Khartoum, P.O. Box 321, Khartoum 11115, Sudan

PACS 03.65.Ta – Foundations of quantum mechanics

Abstract – An extension to the Dirac current and continuity equations are proposed. A general form for the current densities associated with the particle motion is obtained. One current density is shown to be over a plane (surface), and the other one acts perpendicular to it. A generalized set of continuity equations connecting the currents and charge densities are found, and a transformation under which these equations are invariant is proposed. This set of equations is shown to be analogous to the Maxwell's equations. In an analogy with electromagnetism, two electromotive-like forces are found that causes the motion of the particle. The charge density, in addition to the two currents, are found to obey a wave equation traveling at the speed of light. The application of the new transformation in Maxwell's equations induces electric charge and current densities. A drag-like force is found to be associated with the motion of a Dirac's particle. A new gauge-like transformation of the quaternionic field is proposed with a new definition of the electromagnetic field.

Introduction. – Any closed electric system is known to comply with the continuity equation that manifests the charge conservation of the system. A current in the physical world is generally an expression for the flow of a conserved quantity, like charge, baryon number, lepton number, etc. The electron, as described by the Schrodinger wave equation satisfies the continuity equation. Similarly, in the relativistic case, the electron is described by the Dirac equation, where a relativistic analog of the continuity equation is shown to govern its motion [1]. At the same time, the propagation of sound in a medium is also described by a continuity equation. In fluid dynamics, the continuity equation expresses conservation of mass. Therefore, the continuity equation constitutes a fundamental integral equation in physical worlds. It also expresses the conservation of energy of a system.

In a relativistic world, the current density is expressed in terms of a 4-vector. This 4-vector is defined in terms of a temporal part in addition to a spatial vector part. The ordinary vector density is represented by the spatial vector part, whereas the density is expressed by the fourth temporal part of the 4-vector. In some representations, the scalar part is defined by an imaginary number, while the spatial part is a real number. To generalize this representation, we employ biquaternions, where the two numbers are generally complex [2]. Though

[a] arbab.ibrahim@gmail.com

the electromagnetic field is represented by a complex number, the current density was not. As quantum particle the is governed by Dirac's equation, whereas a charge is described by Maxwell's equations. Thus mass and charge must be coherently described by a system of equations that reflects the intimate relation between them.

We found the quaternions to be the right mathematical construct that manifests this intimacy. We propose a new definition of the electric and magnetic potentials that realizes this relation between charge and mass description. Note that the particle dynamics is described by the force while the field dynamics is described by Maxwell's equations. However, Dirac's equation describes the particle dynamics. Thus, the three representations must be unified in a single description. The particle and field dynamics are merged. The new definition of the electric and magnetic fields utilizes the Dirac matrices and the scalar and vector potentials. This is unlike the standard definition where the electric and magnetic fields are defined via the derivatives of these scalars. We would like here to show first the correspondence between Maxwell's and Dirac's description of the electron's fields. We then investigate the consequence of such correspondence.

We aim here to make this representation and explore the physical significance of these complex quantities. We then derive the continuity equation that connects the temporal and spatial variations of these quantities, in the framework of the Dirac equation. We also want to see how the de Broglie wave associated with the relativistic Dirac's particle is associated with it. Recall that the electromagnetic wave arising from varying electric and magnetic fields is known to propagate in a direction perpendicular to the plane spanned by the electric and magnetic fields.

To maintain a physical signature of the different components of the current density, we resort to the Dirac-Maxwell analogy, we recently established. The continuity equation is known to be a single scalar equation. However, in the present formalism, a generalized set of continuity equations are obtained and their solution is found. These are shown to correspond to two scalar equations and two vector equations. They have a common structure as that of the four Maxwell's equations. We have derived before, the Maxwell's equations from the quaternionic Dirac equation [3, 4].

The quaternionic Dirac and continuity equations. – The Dirac equation is a relativistic and covariant equation that best describes the motion of an electron that is expressed as [1]

$$i\hbar\gamma^\mu\partial_\mu\psi - mc\,\psi = 0\,, \qquad p^\mu\gamma_\mu\psi = mc\,\psi\,, \qquad i\hbar\frac{\partial\psi}{\partial t} = (c\vec{\alpha}\cdot\vec{p} + \beta mc^2)\,\psi\,, \tag{1}$$

where ψ is known as the Dirac spinor (4×1 column matrix), and γ^μ are the 4×4 - Dirac matrices equipped with some algebra [1]. The zero-component is denoted by $\gamma^0 = \beta$, and the spatial components by $\vec{\gamma} = \beta\vec{\alpha}$.

Equation (1) can be seen as a momentum eigenvalue equation, where the momentum operator 4-vector is $p_\mu = i\hbar\partial_\mu$. The continuity equation that guarantees the conservation of electric charge is expressed, in terms of the Dirac spinor, as [1]

$$\partial_\mu J^\mu = 0\,, \qquad J^\mu = \bar{\psi}\gamma^\mu\psi\,. \tag{2}$$

In special relativity, the current density is defined as

$$J^\mu = (c\rho\,, \vec{J}\,)\,. \tag{3}$$

It is shown recently that a quaternionic Dirac equation is rich of information [3–5]. To this aim, we express it in the quaternionic form as

$$\tilde{P}\tilde{\gamma}\tilde{\Psi} = mc\tilde{\Psi}\,, \tag{4}$$

where

$$\tilde{P} = \left(\frac{i}{c}\,E\,, \vec{p}\right)\,, \qquad \tilde{\gamma} = (i\beta\,, \vec{\gamma}\,)\,, \qquad \tilde{\Psi} = \left(\frac{i}{c}\,\psi_0,\, \vec{\psi}\right)\,, \tag{5}$$

which generalize the ordinary definitions of the above quantities. Here $\tilde{\Psi}$ is the quaternionic wavefunction embodying the basic fields describing the quantum particle. It is found that Eq.(4) yields a Maxwellian form for the quaternionic field. That equation reveals that the concept of the monopole is inherent in the resulting equations, not as the case with the ordinary Maxwell's equations [6]. However, if we consider the mass in Eq.(Eq.(4) to be pure imaginary ($\pm\, im$), then one finds

$$\vec{\nabla}\cdot\vec{E}_d = \frac{\rho_d}{\varepsilon_0} - \frac{\partial\Lambda_d}{\partial t}\,, \qquad \vec{\nabla}\cdot\vec{B}_d = 0\,, \tag{6}$$

and

$$\vec{\nabla}\times\vec{E}_d = -\frac{\partial\vec{B}_d}{\partial t}\,, \qquad \vec{\nabla}\times\vec{B}_d = \frac{1}{c^2}\frac{\partial\vec{E}_d}{\partial t} + \mu_0\vec{J}_d + \vec{\nabla}\Lambda_d\,, \tag{7}$$

where

$$\vec{E}_d = -(c\beta\vec{\psi} + \vec{\gamma}\,\psi_0)\,, \qquad \vec{B}_d = \vec{\gamma}\times\vec{\psi}\,, \qquad \rho_d = \mp\frac{\varepsilon_0 mc}{\hbar}\,\psi_0\,, \qquad \vec{J}_d = \mp\frac{mc}{\mu_0\hbar}\,\vec{\psi}\,, \qquad \Lambda_d = \vec{\gamma}\cdot\vec{\psi} + \frac{\beta}{c}\,\psi_0\,. \tag{8}$$

We better call $\vec{E}_d$ and $\vec{B}_d$ the Dirac inertial and magnetic fields, that are due to the mass of the particle, in comparison with Maxwell's fields that are due to the charge of the particle. We see here that ψ_0 and $\vec{\psi}$ are analogous of the electromagnetic potentials, φ and $\vec{A}$, respectively. It is worth to mention here that some authors treat $\vec{A}$ as the photon field [7,8]. Using Eq.(8), one can deduce the relation, $\vec{J}_d = \frac{c^2\rho_d\vec{\psi}}{\psi_0}$ that is analogous to the

Einstein relativistic relation $\vec{v}=\frac{c^2\vec{p}}{E}$, suggesting that $\vec{\psi}$ can be connected with the particle linear momentum, and ψ_0 with its energy.

It is interesting to connect the $\mp$ sign to the particle and antiparticle cases [3]. Remarkably, the particle behavior due to its mass is compatible with that due to its charge. This is anticipated since the charge and the mass are intimately connected with the particle. It is also recently found that the quantum behavior of electrons in the transmission line is the same as their electric behavior [5]. It is apparent from Eq.(8) that the matter fields are the potentials themselves, Maxwell's fields are defined by the derivatives of the potentials.

We envisage here the analogy between matter (Dirac) field and charge (Maxwell) fields. While the Maxwell's fields are defined by space and time derivatives of the vector and scalar potentials, the Dirac fields are linear combinations of the basic fields $(\psi_0\,,\vec{\psi}\,)$. Apart from the subscript "d", the system of above equations is but the modified Maxwell's equations when Lorenz gauge is relaxed [9]. Moreover, in Dirac paradigm, the source of the matter fields is the particle mass, while in Maxwell's, the sources are the charges and currents. Note that as the case for Maxwell's fields, the Dirac fields are perpendicular to each other, *i.e.*, $\vec{E}_d\cdot\vec{B}_d=0\,.$ A massless particle has no matter source, *i.e.*, $\rho_d=0$ and $\vec{J}_d=0\,.$ Equation (8) reveals the interesting relations

$$\vec{J}_d\times\vec{E}_d=\rho_d\,c^2\vec{B}_d\,,\qquad \vec{E}_d=-c\beta\vec{\gamma}\times\vec{B}_d+c\beta\vec{\gamma}\Lambda_d\,,\qquad \vec{B}_d=\frac{\beta\vec{\gamma}}{c}\times\vec{E}_d\,,\qquad c\beta\vec{\gamma}\cdot\vec{E}_d=c^2\Lambda_d\,. \tag{9}$$

The fields in Eq.(9) are the fields produced by the moving quantum particle. Moreover, one also finds

$$\vec{E}_d\times\vec{B}_d=(\psi^2-\frac{\psi_0^2}{c^2})\,c\beta\vec{\gamma}-c\Lambda_d\beta\,\vec{\psi}\,. \tag{10}$$

Because of the Λ_d term, the electric and magnetic fields, $\vec{E}$, $\vec{B}$, are not perpendicular to the particle velocity, $\vec{v}=c\beta\vec{\gamma}$, as anticipated in the ordinary case (for Maxwell's fields); and therefore, these vectors do not form a set of mutually orthogonal vectors. It is thus very urging to seek the physical account of the scalar field, Λ_d.

For a particular case, $\vec{J}_d=\rho_d\,\vec{v}$, one obtains the magnetic field of a moving charge, *viz.*, $\vec{B}_d=\frac{\vec{v}}{c^2}\times\vec{E}_d$ and $\vec{E}_d=-\vec{v}\times\vec{B}+\vec{v}\,\Lambda_d\,.$ In Dirac's theory one has $\vec{v}=c\vec{\alpha}=c\beta\vec{\gamma}$. It is interesting to recall that Eq.(9) is obtained by considering massive photons to be described by the vector potential [8]. It is thus quite remarkable that the mass field and charge field are governed by the same equations. This implies that Dirac and Maxwell's equations describe but the same physical entity of the particle. The matter electromagnetic field of the particle obeys the same equations as that due to particle charge. It is thus important to describe the Dirac's equation by matter fields rather than by wavefunctions. The question that arises is whether one can reexpress the electromagnetic fields in Maxwell's theory in terms of a linear combination of the scalar and vector potentials rather than the derivatives of these potentials following the above formulation. To realize this, we use the ansatz, $\vec{\psi} = \kappa\vec{A}$ and $\psi_0 = \kappa\varphi$, where κ is some constant that can be chosen as the Compton wavelength of the particle. Therefore, one will define the electromagnetic field and the scalar Λ_d, as,

$$\vec{E}_m = -\kappa\,(c\beta\vec{A} + \vec{\gamma}\,\varphi)\,, \qquad \vec{B}_m = \kappa\,\vec{\gamma}\times\vec{A}\,, \qquad \Lambda_m = \kappa\,(\vec{\gamma}\cdot\vec{A} + \frac{\beta}{c}\,\varphi)\,. \tag{11}$$

Henceforth, one unifies Maxwell and Dirac equations instead of the standard formulation adopted by quantum electrodynamics. The above expression entitles the electromagnetic field to be an operator since $\vec{\gamma}$ and β are matrices. The new gauge transformation associated with the fields in Eq.(11) is

$$\vec{A}\,' = \vec{A} \pm \frac{\varphi}{c}\,\vec{\gamma}\,, \qquad \varphi\,' = (1\pm\beta)\,\varphi\ ,$$

under which Eqs.(6) - (8) are invariant. It is analogous to Lorentz boost of the coordinates.

Let us now define the ordinary electromagnetic field as

$$\vec{E} = -\vec{\nabla}\varphi - \frac{\partial\vec{A}}{\partial t}\,, \qquad \vec{B} = \vec{\nabla}\times\vec{A}\,. \tag{12}$$

We are here to relate the two fields, $\vec{E}_m$ & $\vec{B}_m$ with $\vec{E}$ & $\vec{B}$.

Now Faraday's equation will be expressed by

$$\vec{\nabla}\times\vec{E}_m = -\frac{\partial\vec{B}_m}{\partial t}\,, \tag{13}$$

which upon using Eqs.(11) yields the relation

$$\vec{B} = -\frac{\beta\vec{\gamma}}{c}\times\vec{E}\,. \tag{14}$$

Now the modified Ampere's equation can be written as

$$\vec{\nabla}\times\vec{B}_m = \frac{1}{c^2}\frac{\partial\vec{E}_m}{\partial t} + \mu_0\vec{J}_m - \vec{\nabla}\Lambda_m\,, \tag{15}$$

which upon using Eqs.(11) yields the relations

$$\vec{E} = c\beta\vec{\gamma}\times\vec{B} - \mu_0 c\beta\,\kappa\vec{J} + c\beta\vec{\gamma}\,\Lambda\,, \qquad \vec{J} = \frac{1}{\mu_0\kappa}\,\vec{\gamma}\times\vec{B} - \frac{\beta}{\mu_0 c\,\kappa}\,\vec{E} + \frac{\vec{\gamma}}{\mu_0\kappa}\,\Lambda \tag{16}$$

where

$$\Lambda = \frac{1}{c^2}\frac{\partial\varphi}{\partial t} + \vec{\nabla}\cdot\vec{A}\,, \tag{17}$$

where Λ is a measure of the violation of the Lorenz gauge condition of electromagnetism; and we set $\vec{J}_m = \kappa\,\vec{J}$ and $\rho_m = \kappa\,\rho$. The electric field in Eq.(16) is that due to a particle and its antiparticle with spin sates (up and down).

We call this electric field the spinorial electric field. Equation (16) can be seen as defining a modified Ohm's law that appears frequently in plasma hydrodynamics, where the second term in Eq.(16) is a conduction current in conductors, with $\vec{J} = \sigma\,\vec{E}$, where $\sigma = (\mu_0 c\kappa)^{-1}$. Equation (16) can be expressed as

$$\vec{J} = \pm\,\sigma\left(\vec{v}\times\vec{B} - \vec{E} + \vec{v}\,\Lambda\right).$$

Recall that the eigenvalues of β are ±1 representing a particle and its antiparticle with spin-up and spin-down states. Finally, applying Eq.(8) in the modified Gauss's and the divergence laws

$$\vec{\nabla}\cdot\vec{E}_m = \frac{\partial\Lambda_m}{\partial t} + \frac{\rho_m}{\varepsilon_0}\,, \qquad \vec{\nabla}\cdot\vec{B}_m = 0\,, \tag{18}$$

yield

$$c\beta\vec{\gamma}\cdot\vec{B} = 0\,, \qquad c\beta\vec{\gamma}\cdot\vec{E} = \frac{c\beta\kappa}{\varepsilon_0}\,\rho + c^2\Lambda\,. \tag{19}$$

These equations were obtained in [8] when considering the photon wavefunction to be $\vec{A}$ and φ. It is interesting that the proposal in Eqs.(11) yields the particle electrodynamics. Recall that the magnetic field in Eqs.(14) and the electric field in Eqs.(16) are those due to a moving charge with velocity, $\vec{v} = c\beta\vec{\gamma}$. The power delivered to the particle can be obtained from Eqs.(19) using the definition, $P = q\vec{v}\cdot\vec{E}$. It is interesting to see that the field and particle dynamics are intimately connected. In vacuum, ρ and $\vec{J} = 0$. It is remarkable that the particle dynamics, Eqs.(14), (16) and (19), is derived from the modified Maxwell's field equations, Eq.(13), (15) and (18).

Equations (6) and (7) can be manipulated to yield the wave equations

$$\frac{1}{c^2}\frac{\partial^2\vec{E}_d}{\partial t^2} - \nabla^2\vec{E}_d = -\frac{1}{\varepsilon_0}\left(\vec{\nabla}\rho_d + \frac{1}{c^2}\frac{\partial\vec{J}_d}{\partial t}\right), \qquad \frac{1}{c^2}\frac{\partial^2\vec{B}_d}{\partial t^2} - \nabla^2\vec{B}_d = \mu_0\vec{\nabla}\times\vec{J}_d\,, \qquad \frac{1}{c^2}\frac{\partial^2\Lambda_d}{\partial t^2} - \nabla^2\Lambda_d = 0\,.$$

Let us now consider the force acting on the mass using Lorentz force as

$$\vec{F}_d = m(\vec{E}_d + \vec{v}\times\vec{B}_d)\,. \tag{20}$$

Equations (9) and (10) express the particle electrodynamics characteristics. Using Eq.(8), the particle's force and acceleration can be expressed as

$$\vec{F}_d = -m\Lambda_d\,\vec{v}\,, \qquad \vec{a}_d = -\Lambda_d\,c\beta\,\vec{\gamma}\,. \tag{21}$$

The above force is a drag (viscous) force type. It seems as if the particle is moving in a pre-existing fluid. Such a fluid was named the Ether that was put forward to justify the motion of the electromagnetic wave and later on ruled out. It is interesting to see the Ether is reflected in the motion of massive object, *e.g.*, an electron. Massless particle feels no force according to Eq.(21), however. The viscous force above will disappear if $\Lambda_d = 0$, and will move with constant velocity. Equation (21) is a force equation consisting of four equations describing the force experienced by a moving particle/antiparticle. The power delivered to the particle is given by, $P_d = \vec{F}_d \cdot \vec{v}$, where $\vec{v} = c\beta\,\vec{\gamma}$, which upon using Eq.(8) yields

$$P_d = mc^2\Lambda_d\,. \tag{22}$$

A gauge-like transformation connected with Eqs.(6)- (8) can be found by expressing

$$\vec{\psi}\,' = \vec{\psi} \pm \frac{\psi_0}{c}\,\vec{\gamma}\,, \qquad\qquad \psi_0' = (1 \pm \beta)\,\psi_0\,, \tag{23}$$

under which all expressions in Eq.(8) are invariant, *i.e.*, $\vec{E}_d\,' = \vec{E}_d$, $\vec{B}_d\,' = \vec{B}_d$ and $\Lambda_d\,' = \Lambda_d$, when $m = 0$. It is apparent that a massive particle spoils the new gauge transformation. It is analogous to the case of massive photon in electrodynamics.

The energy conservation equation associated with the system of equations, Eqs.(6) - (8), is given by

$$\vec{\nabla}\cdot\vec{S}_d + \frac{\partial u_d}{\partial t} = -\vec{J}_d\cdot\vec{E}_d + c^2\Lambda_d\,\rho_d\,, \tag{24}$$

where

$$\vec{S}_d = \mu_0^{-1}(\vec{E}_d \times \vec{B}_d - \Lambda_d\,\vec{E}_d)\,, \qquad\qquad u_d = \frac{\varepsilon_0 E_d^2}{2} + \frac{B_d^2}{2\mu_0} + \frac{\Lambda_d^2}{2\mu_0}\,. \tag{25}$$

We notice here $\vec{S}_d$ is the energy flux flow of the matter field, and u_d is its energy density. Using Eq.(8), one finds

$$\vec{S}_d = \mu_0^{-1}\left(2\psi_0\,\Lambda_d - c\beta(\psi^2 + \frac{\psi_0^2}{c^2})\right)\vec{\gamma}\,, \qquad\qquad \vec{S}_d = \mu_0^{-1}\left(2\frac{\psi_0}{c}\,\Lambda_d\beta - (\psi^2 + \frac{\psi_0^2}{c^2})\right)\vec{v}\,. \tag{26}$$

Using Eq.(8), one finds

$$\frac{E_d^2}{c^2} = \psi^2 - \frac{\psi_0^2}{c^2}\,, \qquad B_d^2 = -\psi^2 - (\vec{\gamma}\cdot\vec{\psi})^2\,, \qquad \Lambda_d^2 = (\vec{\gamma}\cdot\vec{\psi})^2 + \frac{\psi_0^2}{c^2}\,, \qquad u_d = 0\,.$$

It is interesting to notice that the energy flows along the velocity direction, recalling that in Dirac's theory $\vec{v} = c\beta\,\vec{\gamma}$. Despite this the particle field carries no energy since $u_d = 0$. Therefore, the energy is utterly carried by the particle.

The standard electrodynamics is obtained if we set $\Lambda = 0$, Eq.(17), where the Lorenz gauge condition is satisfied. However, if we now set $\Lambda_d = 0$ for particle dynamics, then $\vec{S}_d \propto \vec{v}$, implying that the energy flows along the particle velocity direction.

The energy lost by the matter wave (power density), which is the right hand-side of Eq.(24), can be expressed as, using Eq.(8),

$$p_d = \mp\frac{mc}{\mu_0\hbar}\,(c\beta\psi^2 + \Lambda_d\,\psi_0)\,.$$

It is interesting to remark that Eq.(24) and (25) are invariant under the gauge-like transformation expressed in Eq.(23). Hence, the above formulation represents an expression of rewriting the equations of the quantum mechanics, instead of the wavefunction, in terms of matter fields that become indistinguishable from the Maxwellian formulation. Therefore, quantum mechanics in this formulation is no longer a probabilistic theory.

We would like here to explore the Dirac continuity equation aiming at deriving a more general (complex) form of the charge and current densities, compared to the conventional ones defined in Eqs.(2) and (3).

Quaternionic current density. – The biquaternionic Dirac current density will generalize the ordinary Dirac current density to be complex. Inspired by Eq.(2), this can be expressed as a biquaternionic current density

$$\tilde{J}=\tilde{\Psi}^*\tilde{\gamma}\tilde{\Psi}\,, \tag{27}$$

which can be written in the standard form as

$$\tilde{J}=(ic\rho_D\,,\vec{J}_D\,)\,, \tag{28}$$

where $\vec{J}_D$ can be complex. Expanding Eq.(27) using Eq.(5) yields

$$\rho_D=\frac{\beta}{c}\left(\psi^2-\frac{\psi_0^2}{c^2}\right)\,,\qquad\qquad \vec{J}_D=\frac{2i}{c}\,\psi_0(\vec{\gamma}\times\vec{\psi})-\left(\frac{\psi_0^2}{c^2}+\psi^2\right)\vec{\gamma}+2(\vec{\gamma}\cdot\vec{\psi})\,\vec{\psi}\,. \tag{29}$$

The generalized Dirac current density $\vec{J}_D$ can be expressed as, $\vec{J}_D=\vec{J}_r+i\,\vec{J}_t$, where

$$\vec{J}_r=2(\vec{\gamma}\cdot\vec{\psi})\,\vec{\psi}-\left(\frac{\psi_0^2}{c^2}+\psi^2\right)\vec{\gamma}\,,\qquad\qquad \vec{J}_t=\frac{2}{c}\,\psi_0(\vec{\gamma}\times\vec{\psi})\,. \tag{30}$$

It is remarkable that to see that the real current flows along the $\vec{\psi}-\vec{\gamma}$ plane, while the imaginary current flows along the direction perpendicular to $\vec{\psi}$ and $\vec{\gamma}$, since $\vec{J}_r\cdot\vec{J}_t=0$. We call the latter current, a *transverse current*, while the former a *horizontal current*. The horizontal current reveals that the Dirac particle doesn't travel as a single particle but rather like a fluid. The transverse current $\vec{J}_t$ mimics the magnetic field in the electromagnetic wave that is perpendicular to the plane containing the electric field and the direction of energy flow. The direction of the velocity of the Dirac particle is along the $\vec{\gamma}$ direction. One can now make a comparison between the matter wave flux density and the particle current density (Eqs.(26) and (30)). This comparison will shed light on how the wave-particle duality works.

Therefore, in analogy with the electromagnetic theory, our present theory states that the wave vector, $\vec{\psi}$, is analogous to the electric field, the transverse current, $\vec{J}_t$, is analogous to the magnetic field, and $\vec{\gamma}$ is analogous to the wave velocity. It is pertinent to see that the simple relation $\vec{J}=\rho\vec{v}$ no longer exits in this formalism.

Biquaternionic continuity equation. – The biquaternionic current in Eqs.(27) and (28) is governed by the biquaternionic continuity equation [2]

$$\tilde{\nabla}\tilde{J} = 0\,. \tag{31}$$

Expanding Eq.(31), and equating the real and imaginary parts for the scalar and vector parts of the resulting equations to zero, we obtain

$$\vec{\nabla}\cdot\vec{J}_r + \frac{\partial\rho}{\partial t} = 0\,,\qquad \vec{\nabla}\cdot\vec{J}_t = 0\,,\qquad \frac{\partial\vec{J}_r}{\partial t} + c^2\vec{\nabla}\rho + c\vec{\nabla}\times\vec{J}_t = 0\,,\qquad \vec{\nabla}\times\vec{J}_r - \frac{1}{c}\frac{\partial\vec{J}_t}{\partial t} = 0\,, \tag{32}$$

where $\rho = \rho_D$. We thus see from the above equation that the two currents are coupled to each other in a similar way the electric and magnetic fields couple in Maxwell's equations. We already had an experience of the effect of the temporal variation of the counterpart of a give quantity described by some system of equations. Of such currents, is the displacement Maxwell's current. Such a system of equations will find applications in the field of hydrodynamics and plasma physics. The transverse current tends to lift the object (fluid) upward. London modelled the phenomenon of superconductivity as due to the flow of two currents; the ordinary electric current and a superelectric current that floats over it [10]. Thus, a question could arise if one can associate this supercurrent to the transverse current above. Equation (32) is solved to give

$$\frac{1}{c^2}\frac{\partial^2\rho}{\partial t^2} - \nabla^2\rho = 0\,,\qquad \frac{1}{c^2}\frac{\partial^2\vec{J}_r}{\partial t^2} - \nabla^2\vec{J}_r = 0\,,\qquad \frac{1}{c^2}\frac{\partial^2\vec{J}_t}{\partial t^2} - \nabla^2\vec{J}_t = 0\,. \tag{33}$$

Hence, $\rho\,,\vec{J}_r\,,\vec{J}_t\,,$ satisfy the wave equation travelling at the speed of light. It is thus interesting that any change in the charge of current densities in a place, will be immediately transmitted as a wave. However, using the ordinary continuity equation, one can't deduce that the current is a wave traveling at the speed of light.

The system of equations in Eq.(32) reduces to our generalized continuity equation [2], when $\vec{J}_t = 0$; that occurs, either when $\psi_0 = 0$, or $\vec{\psi} = 0$, referring to Eq.(30). In this special case, one has $\vec{J}_r = 2(\vec{\gamma}\cdot\vec{\psi})\,\vec{\psi} - \rho\,\vec{v}$, and $\vec{J}_r = -\left(\frac{\psi_0^2}{c^2}\right)\vec{\gamma} = \rho\,\vec{v}$, respectively. A third possibility exists, when $\vec{\gamma}$ is parallel to $\vec{\psi}$, for which $\vec{J}_r = -\rho\,\vec{v}$, by virtue of Eqs.(29) and (30), and the definition $\vec{\alpha} = \beta\vec{\gamma}$, where $\vec{v} = c\beta\vec{\gamma}$.

Now the last equation in Eq.(32) can be manipulated to give

$$\epsilon_t = -L\frac{\partial I_t}{\partial t}\,,\qquad \Rightarrow\qquad \epsilon_t = -\int \vec{E}_r\cdot d\vec{\ell}\,,\qquad \vec{E}_r = cL\,\vec{J}_r\,, \tag{34}$$

where L denotes some inertial inductance, and $\vec{E}_r$ some electric-like field. The kinetic inductance for a conductor of a cross-sectional area, A and length ℓ, is defined as

$$L_k = \frac{m}{ne^2}\frac{\ell}{A}\,,$$

where e, and n are the electric charge, and carrier concentration in the conductor. Similarly, the third equation in Eq.(32) can be manipulated to yield

$$\epsilon_r = -L\frac{\partial I_r}{\partial t}\,, \qquad \Rightarrow \qquad \epsilon_r = \int \vec{E}_t \cdot d\vec{\ell}\,, \qquad \vec{E}_t = cL\,\vec{J}_t\ , \tag{35}$$

which allows us to express the electromotive-like force as a complex number in the form, $\epsilon_D = \epsilon_r + i\,\epsilon_t$.

Equations (34) and (35) show that the electric-like fields, $\vec{E}_r$ and $\vec{E}_t$, are measures of the horizontal (surface) and transverse current densities. They can be expressed as

$$\vec{J}_r = \sigma_r \vec{E}_r\ , \qquad \vec{J}_t = \sigma_c \vec{E}_t\ , \qquad \sigma_r = \sigma_c = \frac{1}{Lc}\,, \tag{36}$$

where σ_r and σ_c are the horizontal and transverse electric-like conductivities.

The time change of the horizontal current induces a circulation in the transverse current, and vice versa. But the two effects produce opposite effects. We may treat as that one which lines spread over the plane $\vec{\psi} - \vec{\gamma}$. These two new quantities have to be investigated. It is interesting to see from Eqs.(34) and (35) that an electromotive-like forces (ϵ_r, ϵ_c) arise due to temporal variation of the transverse and horizontal currents that push the particle to move.

London's equations of superconductivity. London's described the superconductor by the two equations [10]

$$\frac{\partial \vec{J}}{\partial t} + c^2\vec{\nabla}\rho = \frac{ne^2}{m}\,\vec{E}\,, \qquad \vec{\nabla}\times\vec{J} = -\frac{ne^2}{m}\,\vec{B}\,. \tag{37}$$

Comparing Eq.(37) with Eq.(32) yields

$$\frac{1}{c}\frac{\partial \vec{J}_t}{\partial t} = -\frac{ne^2}{m}\,\vec{B}\,, \qquad c\vec{\nabla}\times\vec{J}_t = -\frac{ne^2}{m}\,\vec{E}\,, \tag{38}$$

which define the transverse current in a superconductor. It is interesting to notice that the temporal variation of the transverse current gives rise to a magnetic field, whereas the spatial variation produces and electric field. This is complimentary to the effect of the ordinary current. While the electric and magnetic fields in Eq.(37) salsify the Klein-Gordon equation, they satisfy the wave equation with zero mass in Eq.(38), where the ordinary charge and current densities vanish. Therefore, inside a superconductor the transverse current exists when the ordinary current is absent. Moreover, Eqs.(37) and (38) can be seen as dual to each other when $\vec{\nabla}\rho = 0$. A dual superconductor is proposed to s to model the confinement of Quantum ChromoDynamics (QCD). It is speculated that the QCD vacuum can be described in terms of a Landau-Ginzburg model of a dual superconductor [11]. To remedy Eq.(38)

an additional magnetic charge gradient can be added to its left hand-side. Hence, the existence of the transverse current is so fundamental to bring the duality.

In terms of the scalar and vector potentials, Eqs.(38) and (32) reveal that these potentials, in addition to the transverse current, obey a wave equation traveling at the speed of light. Thus, the transverse current is the current that is established when the normal current and charge densities are absent.

Energy and momentum conservation equations. The set of equations in Eq.(32) can be connected with an energy conservation equation that can be obtained from Eq.(32) as

$$\frac{\partial u}{\partial t}+\vec{\nabla}\cdot\vec{S}=0\,, \tag{39}$$

where

$$u=\frac{J_t^2}{2c}+\frac{J_r^2}{2c}+\frac{c\rho^2}{2}\,,\qquad\qquad \vec{S}=\vec{J}_t\times\vec{J}_r+c\rho\,\vec{J}_r\,. \tag{40}$$

Note that Eq.(37) can also be seen as representing a continuity equation too. The momentum conservation equation can be obtained from Eq.(32) where

$$\frac{\partial g_i}{\partial t}+\partial_j T_{ij}-\vec{\nabla}\times(c^2\rho\vec{J}_r)=0\,, \tag{41}$$

where

$$\vec{g}=\vec{J}_t\times\vec{J}_r-c\rho\vec{J}_r\,,\qquad\qquad T_{ij}=\frac{c}{2}\left(J_t^2+J_r^2-c^2\rho^2\right)\delta_{ij}-c\left(J_{r\,i}J_{r\,j}+J_{t\,i}J_{t\,j}\right)-\epsilon_{ijk}c^2\rho J_{r\,k}=0\,. \tag{42}$$

Note that T_{ij} can be see as a stress tensor, u as an energy density, $\vec{S}$ as a Poynting vector, and $\vec{g}$ as a momentum density vector. In such conditions, the last term on the left hand-side of Eq.(41), will represent a force density acting on the fluid. It is a matter of interest if a fluid (with two components) satisfying the above equations really exists. The last term in Eq.(41) $\vec{\tau}=c^2\rho\vec{J}_r$ can be seen as a torque density acting on the fluid.

Current-charge densities transformation. It is very interesting to observe that Eq.(32) is invariant under the charge-current densities transformation

$$\vec{J}_t\,'=\vec{J}_t+\vec{\nabla}\times\vec{g}\,,\qquad \vec{J}_r\,'=\vec{J}_r+\frac{1}{c}\frac{\partial\vec{g}}{\partial t}\,,\qquad \rho\,'=\rho-\frac{1}{c}\vec{\nabla}\cdot\vec{g}\,, \tag{43}$$

where $\vec{g}$ is some vector function obeying a wave equation traveling at the speed of light, *i.e.*,

$$\frac{1}{c^2}\frac{\partial^2\vec{g}}{\partial t^2}-\nabla^2\vec{g}=0\,. \tag{44}$$

Notice that the transformations of ρ and $\vec{J}_r$ are somehow analogous to the gauge transformations of the electromagnetic fields. Looking at Eq.(32) with some scrutiny reveals that they are analogous to Maxwell's equations.

The transformation in Eq.(43) suggests induced charge and current densities given respectively by

$$\vec{J}_i = \frac{1}{c}\frac{\partial \vec{g}}{\partial t}\,, \qquad\qquad \rho_i = -\frac{1}{c}\vec{\nabla}\cdot\vec{g}\,. \tag{45}$$

Plugging these quantities in Maxwell's equations, in vacuum, gives rise to a displacement vector given by

$$\vec{D} = \varepsilon_0\vec{E} + \frac{\vec{g}}{c}\,, \tag{46}$$

which when compared with the standard case, yields a polarization vector, $\vec{P} = \vec{g}/c$. Therefore, the transformation in Eq.(43) is such that it induces a polarization terms in the Maxwell's equations.

Let us now use Eqs.(34) and (35) to rewrite Eq.(32) as

$$\vec{\nabla}\cdot\vec{E}_r = -\frac{\partial(Lc\rho)}{\partial t}\,, \qquad \vec{\nabla}\cdot\vec{B}_r = 0\,, \qquad \vec{\nabla}\times\vec{B}_r = -\frac{1}{c^2}\frac{\partial \vec{E}_r}{\partial t} - \vec{\nabla}(Lc\rho)\,, \qquad \vec{\nabla}\times\vec{E}_r - \frac{\partial \vec{B}_r}{\partial t} = 0\,, \tag{47}$$

where we defined

$$\vec{B}_r = L\vec{J}_t\,. \tag{48}$$

The above definition of the magnetic field is welcomed since $\vec{B}_r$ is perpendicular to $\vec{E}_r$ which is the case for Maxwell's electromagnetic field. It pertinent to mention that when the subscript "r" in Eq.(47) is dropped, it yields the modified Maxwell's equations [9]. Equation (47) can be compared with the electrodynamics vacuum equations, Eqs.(13), (15) and (18).

Interestingly, the set of equations in Eq.(47) is but the Maxwell's equations of the electromagnetic field with an effective charge and current densities give respectively by

$$\rho_f = \frac{1}{c^2}\frac{\partial \Lambda}{\partial t}\,, \qquad\qquad \vec{J}_f = -\vec{\nabla}\Lambda\,, \qquad\qquad \Lambda = \frac{Lc}{\mu_0}\rho \tag{49}$$

that satisfy the continuity equation

$$\vec{\nabla}\cdot\vec{J}_f + \frac{\partial \rho_f}{\partial t} = 0\,, \tag{50}$$

by virtue of Eq.(33). It is interesting to see the connection between Eqs.(49) and (45), that allows the case that, $\rho_f = \rho_i$ and $\vec{J}_f = \vec{J}_i$. It is remarkable to see that the Maxwell's equations are derived from the quaternionic Dirac continuity equation. We have derived before, the Maxwell's equations from the quaternionic Dirac equation [3,5]. An electrodynamics of the above form is recently addressed and their consequences are studied [9]. The electric charge and current densities appearing in Eq.(47) are found to be associated with thermoelectricity [12].

Electromagnetic wave equations. From Maxwell's equations, one finds the following wave equations that describe the evolution of the electromagnetic field

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}-\nabla^2\vec{E}=-\mu_0\left(\frac{\partial\vec{J}}{\partial t}+c^2\vec{\nabla}\rho\right), \tag{51}$$

and

$$\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}-\nabla^2\vec{B}=\mu_0\vec{\nabla}\times\vec{J}. \tag{52}$$

Comparing the right hand-sides of Eqs.(51) and (52) with Eq.(32) suggest that one can extend the above wave equations to include the transverse current. Therefore, one obtains

$$\frac{1}{c^2}\frac{\partial^2\vec{E}}{\partial t^2}-\nabla^2\vec{E}=-\mu_0\left(\frac{\partial\vec{J}}{\partial t}+c^2\vec{\nabla}\rho+c\vec{\nabla}\times\vec{J}_t\right), \tag{53}$$

and

$$\frac{1}{c^2}\frac{\partial^2\vec{B}}{\partial t^2}-\nabla^2\vec{B}=\mu_0\left(\vec{\nabla}\times\vec{J}-\frac{1}{c}\frac{\partial\vec{J}_t}{\partial t}\right). \tag{54}$$

Equations (53) and (54) reveal that the temporal and spatial variations are sources of the electromagnetic (wave) field. It is of interest to compare the electromagnetic wave with the gravitational wave that arises not from the temporal and spatial variations of the source, but from the source itself. In Maxwell's theory, the source produces the fields only. The right hand-side of Eqs.(53) and (54) is zero in vacuum.

Concluding remarks. – We extended in this work, the real current density associated with Dirac theory to become complex. In doing so, we have found that the real current acts over a plane containing defined by $\vec{\psi}-\vec{\gamma}$, while the imaginary part represents the current that is perpendicular to the real current. The charge density besides the two currents is shown to obey a wave equation traveling at the speed of light. Two electromotive-like forces are found to exist that are associated with the two currents temporal variations. These two electromotive-like forces are connected. The two currents and the electromotive-like forces are completely determined by $\vec{\psi}$ and $\vec{\gamma}$. Inside the superconductor, the transverse current exists when the normal current is zero. The complex current and the charge densities satisfy a set of four equations. A transformation under which these equations are invariant is found. These equations are analogous to Maxwell's equations of the electromagnetic fields. We thus have derived Maxwell's equations from the quaternionic Dirac continuity equation. A new definition of the electric and magnetic fields that yields Maxwell's equations is proposed. It describes the electromagnetic field due to a particle and its antiparticle with spin-up and spin-down states. This definition gives rise to a modified Ohm's law. The Lorentz force acting on the particle (mass) is of a drag-like force. Further investigations of the physical significance of the proposed quantities are to be explored.

REFERENCES

[1] Bjorken, J. D. and Drell S. D., *Relativistic Quantum Fields*, McGraw-Hill, (1965).
[2] Arbab, A. I., and Widatallah, H. M., *The geenralized continuity equation*, Chinese Phys. Lett. 27, 084703 (2010).
[3] Arbab, A. I., *Derivation of Dirac, Klein-Gordon, Schrodinger, diffusion and quantum heat transport equations from a universal quantum wave equation*, EPL, 92, 40001 (2010).
[4] Arbab, A. I., *The Quaternionic Quantum Mechanics*, Vol. 3, 160 (2011).
[5] Arbab, A. I., *Quantized Maxwell's equations*, Optik, 136, 64 (2017).
[6] Arbab, A. I.,, *Maxwellian quantum mechanics*, Optics 136, 382 (2017).
[7] Bialynicki-Birula, I., *On the wave function of the photon*, Acta Phys. Pol. A 86, 97 (1994).
[8] Arbab, A. I., *Electric and Magnetic Fields Due to Massive Photons and Their Consequences,* Prog. In Electromag. Rese. M, Vol. 34, 153 (2014).
[9] Arbab, A. I., *Extended electrodynamics and its consequences*, Mod. Phys. Lett. B, 31, 1750099 (2017).
[10] London, F., and London, H., *The Electromagnetic Equations of the Supraconductor*, Proc. Roy. Soc. A149, 71 (1935).
[11] Ripka, G., *Dual superconductor models of color confinement* , https://arxiv.org/abs/hep-ph/0310102
[12] Arbab, A. I., *Thermo-electromagnetic transport*, https://arxiv.org/abs/1709.00327.