Exercise 2.3.4 Determine $\sup(A)$ and $\inf(A)$ for the ordered set $(M, \prec)$ and subset A, if these exist.

(d) $A = \{]x, y[\mid -1 < x \leq -\frac{1}{2}, \frac{1}{2} < y \leq 2\}$, wobei $M = \mathfrak{P}(\mathbb{R}), a \prec b :\iff a \subset b$.

Proof. We assume $\sup(A) = B_0 :=]-1, 2[$; B_0 is upper bound of A, because

$$\forall]x, y[\in A :]x, y[\prec B_0 \implies]x, y[\subset B_0.$$

If $B \in M$ is upper bound, then

$$\begin{aligned}\forall]x, y[\in A :]x, y[\prec B &\implies]x, y[\subset B \\ &\implies]-1, 2[\subset B \\ &\implies B_0 \subset B.\end{aligned}$$

Therefore $\sup(A) = B_0$. We assume $\inf(A) = C_0 =]-\frac{1}{2}, \frac{1}{2}[$; C_0 is lower bound of C, because

$$\forall]x, y[\in A : C_0 \prec]x, y[\implies C_0 \subset]x, y[.$$

If $C \in M$ is lower bound, then

$$\begin{aligned}\forall]x, y[\in A : C \prec]x, y[&\implies C \subset]x, y[\\ &\implies C \subset]-\frac{1}{2}, \frac{1}{2}[\\ &\implies C \subset C_0.\end{aligned}$$

Therefore $\inf(A) = C_0$. □