Real Numbers in the Neighborhood of Infinity

Jonathan W. Tooker

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Abstract

We demonstrate the existence of a broad class of real numbers which are not elements of any number field: those in the neighborhood of infinity. After considering the reals and the affinely extended reals, we prove that numbers in the neighborhood of infinity are ordinary real numbers of the type detailed in Euclid's Elements. We show that real numbers in the neighborhood of infinity obey the Archimedes property of real numbers. The main result is an application in complex analysis. We show that the Riemann zeta function has infinitely many non-trivial zeros off the critical line in the neighborhood of infinity.

§1 Introduction

It is the popular theme in modern mathematics to define $\mathbb{R}$ by an algebraic number field approach but here we return to the geometric approach given by the Euclid magnitude [1]. Following Euclid, a real number is the length of a line segment. This length exists completely separately from the axioms of a complete ordered field. We will examine at the end of this paper the Riemann hypothesis which predates Dedekind's work on $\mathbb{R}$ [2] by several years so we are, therefore, perfectly well motivated to eschew the Dedekind cut definition of $\mathbb{R}$ in favor of the Euclid definition. Euclid's Elements is something of a grand canon of mathematics and Euclid's definition of $\mathbb{R}$ dominated the mathematical landscape for thousands of years until the most recent chapter of the history of mathematics began around the turn of the 20th century. Most importantly, Riemann formulated his famous hypothesis during the era in which the Euclid definition of $\mathbb{R}$ was the one in common usage. The axioms of a complete ordered field had not modified the ancient definition of $\mathbb{R}$ at that time, and the Dedekind cut did not exist at that time.

Motivating the present approach, Pugh writes the following in Reference [3].

> "The current teaching trend treats the real number system $\mathbb{R}$ as a given—it is defined axiomatically. Ten or so of its properties are listed, called axioms of a complete ordered field, and the game becomes: deduce its other properties from the axioms. This is something of a fraud, considering that the entire structure of analysis is built on the real number system. For what if a system satisfying the axioms failed to exist? Then one would be studying the empty set!"