Rotor coordinates, vector trigonometry and Kepler-Newton orbits

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1 Introduction

There is *more than one way to skin a cat*, according to a well-known saying. There is also *more than one way to do trigonometry.*

In 2005 I introduced *rational trigonometry* as a simple yet powerful alternative to classical trigonometry, eliminating the need for transcendental functions and calculators, simplifying many problems, and allowing a more careful and logical derivation of Euclidean geometry ([5], see also [7], [9]). Rational trigonometry uses *quadrance* and *spread* instead of *distance* and *angle*, giving a purely algebraic approach to the subject. Hence the theory works over the rational numbers and even over finite fields.

The main laws of rational trigonometry are universal, in that they extend also to other quadratic forms, such as the Einstein form $x^2 + y^2 + z^2 - t^2$ of relativistic geometry. Furthermore, a projective version incorporates *both spherical and hyperbolic geometries* in a novel reformulation, again over a general field (see [6] and [8]).

In this paper I go in a more applied direction, and construct a *vector trigonometry* which is well suited for engineering, surveying and physics applications in the plane. The idea is to replace the usual polar coordinates r and θ of a vector $\mathbf{v}$ with *rotor coordinates* r and h. The quantity r is the usual length, while the *half-turn* h is defined in terms of the rational parametrization of a circle, a notion which can be traced back to Pythagoras. In modern terms the half-turn h is the tangent of half of the polar angle θ, but it is probably better not to dwell on this aspect, as our definition does not require transcendental notions (aside from the square root) or a prior understanding of the concept of angle. We write $\mathbf{v} = |\, r, h\rangle$.

Rotor coordinates replace polar coordinates. While it does not extend to general fields or arbitrary quadratic forms, distance is a primary concept—although angles are not—and orientation can be dealt with directly.

Aspects of this theory will be familiar to many readers, connecting to the Cayley transform of linear algebra, and also to well-known trigonometric formulas. However our avoidance of transcendental functions allows a much wider range of practical computations with vectors, even by hand. The current over-reliance on $90°/45°/45°$ and $90°/60°/30°$ triangles for exercise and exam questions ought to be acknowledged, and overcome.

We can also now address the common misconception that while arithmetical problems demand precision, with geometrical problems a decimal approximation is "good enough". Trigonometry has a rich number–theoretical aspect, in which rational numbers and finite extensions fields (usually quadratic ones) play a key role. This is largely invisible with classical trigonometry.

To show the usefulness of vector trigonometry for geometry, I use it to derive the polynomial relation between the six quadrances determined by four points in the plane, a relation that goes back to Euler's determination of the volume of a tetrahedron in terms of the quadrances of its sides, and which is usually proved using linear algebra.

The main application is to a new treatment of the *Kepler-Newton phenomenon*: planets and comets move in orbits which are conic sections, with the sun at a focus. Understanding the motion of the objects of our solar system is easily the greatest historical problem of the physical sciences, and the remarkable breakthroughs in the 16-th and 17-th centuries due to Galileo, Copernicus, Tycho Brahe, Kepler and Newton are still a cornerstone to the claim that mathematics helps us explain the world in which we live.This paper includes a somewhat new and accessible account that connects naturally with the elementary geometry of conics.

While angles, the transcendental circular functions and their inverse functions play an obvious role in the study of harmonic motion, Fourier series and complex functions, they are not required for trigonometry and vector calculations, where they usually complicate issues and introduce inaccuracies. *Circular harmonic motion* and *affine metrical geometry* are quite different subjects, and do not require a uniform technology. The current practise of lumping these two topics together creates unnecessary difficulties in school mathematics.

The technology developed in this paper, while somewhat elementary, has major implications for both pure and applied mathematics. It provides a more general framework for a wide variety of problems which up to now have mostly been viewed over the real numbers. In particular it opens the way to seeing that a rational number approach can be taken to some theories in geometry and elsewhere in mathematics that up till now have been viewed via transcendental methods. So the connection with computational issues is strengthened.

The rest of this Introduction will give a brief overview of the principal notions of rational trigonometry and then introduce the corresponding ideas for vector trigonometry. Given a vector $\mathbf{v} \equiv (x, y)$, its **quadrance** is the number $Q(\mathbf{v}) \equiv x^2 + y^2$. This is a primary concept. Length, on the other hand, is a secondary concept, the square root of the quadrance. The common practice of referring to quadrance as 'squared length' or 'distance squared' is misguided, as it falsely represents length, or distance, as the primary concept.

Given two vectors $\mathbf{v}_1 \equiv (x_1, y_1)$ and $\mathbf{v}_2 \equiv (x_2, y_2)$, the **spread** between them is the number

$$s(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{(x_1 y_2 - x_2 y_1)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}. \tag{1}$$

Assuming the usual understanding of 'real numbers', the spread $s(\mathbf{v}_1, \mathbf{v}_2)$ is the square of the sine of an angle between $\mathbf{v}_1$ and $\mathbf{v}_2$, but clearly (1) requires no prior definitions of angular measure or circular functions. Furthermore quadrance and spread are valid concepts over a general field F, in particular over the rational numbers. This is one reason why rational trigonometry has its name, another is that the fundamental laws of the subject, relating the quadrances and spreads of a triangle, are expressed by polynomial relations (see [5, Chapter 1]).

If lines l_1 and l_2 have direction vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, then we define $s(l_1, l_2) \equiv s(\mathbf{v}_1, \mathbf{v}_2)$. There are some closely related secondary concepts. The **cross** between the two lines is

$$c(l_1, l_2) \equiv \frac{(x_1 x_2 + y_1 y_2)^2}{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = 1 - s(l_1, l_2)$$

while the **twist** is

$$t(l_1, l_2) \equiv \frac{s(l_1, l_2)}{c(l_1, l_2)} = \frac{(x_1 y_2 - x_2 y_1)^2}{(x_1 x_2 + y_1 y_2)^2}.$$

Since the twist is always a square, we define also the **turn**

$$u(l_1, l_2) \equiv \frac{x_1 y_2 - x_2 y_1}{x_1 x_2 + y_1 y_2}$$

which is an *oriented* quantity; in fact $u(l_2, l_1) = -u(l_1, l_2)$.

In this paper we introduce a *directed* version of these ideas, giving us to define a trigonometry on the vectors themselves, well-suited for practical applications in which direction plays a role. The starting point is the rational parametrization of the unit circle c_U with equation $x^2 + y^2 = 1$:

$$\mathbf{e}(h) \equiv \left(\frac{1 - h^2}{1 + h^2}, \frac{2h}{1 + h^2}\right) \equiv (C(h), S(h)), \tag{2}$$

where h is called the *half-turn* of the vector $\mathbf{v} = \mathbf{e}(h)$, or any positive multiple of $\mathbf{v}$. The half-turn is familiar from calculus, where it appears as the tangent of half of the associated polar angle θ, but the treatment here is independent of classical trigonometry and angles, needing only rational functions and the square root. The *length* $r \equiv |\mathbf{v}| \equiv \sqrt{x^2 + y^2}$ and *half-turn* $h = h(\mathbf{v})$ of a vector $\mathbf{v} \equiv (x, y)$ constitute *rotor coordinates* for $\mathbf{v}$, and we write

$$\mathbf{v} = |\, r, h\rangle .$$

The *Half-turn formula* gives h from the Euclidean coordinates and the length:

$$h = \frac{r - x}{y}.$$

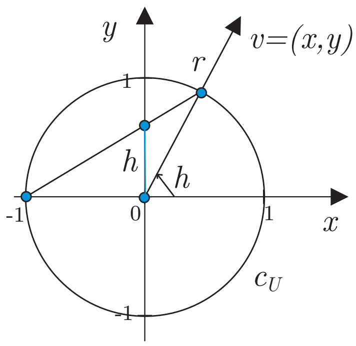

Figure 1: Rotor coordinates $|\, r, h\rangle$ for $\mathbf{v} = (x, y)$

It follows that the half-turn h of a vector $\mathbf{v}$ lies in the same quadratic extension of the field containing x and y as does the length r.

There is a big difference between rotor coordinates r and h, and polar coordinates r and θ. The latter rests on a prior theory of *real numbers*—even if one is working only with rational vectors—since at the very least one needs an understanding of π; a theory of *arc lengths*; and knowledge of the *transcendental circular functions* and *their inverse functions*. These topics are too difficult, and subtle, to be treated carefully and correctly in elementary texts.

Even in more advanced calculus or analysis books, the basic definitions tend to go missing, or rely on a prior theory of Euclidean geometry, which has long been problematic. Logical circularity is, unfortunately, a common feature with current treatments of circular functions. Some of the difficulties were already laid out by Hardy [2] one hundred years ago, but they seem to be mostly ignored these days. Furthermore, polar coordinates have also not been around that long; radian measure for example was introduced only in the late nineteenth century.

With rotor coordinates, algebra replaces analysis. The *circle sum*

$$h_1 \oplus h_2 \equiv \frac{h_1 + h_2}{1 - h_1 h_2}$$

of half-turns replaces the addition of angles, and gives us an algebraic approach to rotations. The extension of the circle sum to more than two inputs is also interesting, for example

$$h_1 \oplus h_2 \oplus h_3 = \frac{h_1 + h_2 + h_3 - h_1 h_2 h_3}{1 - (h_1 h_2 + h_2 h_3 + h_1 h_3)},$$

and the *Circle sum theorem* generalizes this to n values.

Vector trigonometry is then the study of vectors and triangles using rotor coordinates. In applications, the rational numbers may be extended with a value ∞; we also give a projective version of the theory which justifies and explains this. We enlarge the notion of the half-turn of a single vector to define the *relative half-turn* $h(\mathbf{v}_1, \mathbf{v}_2)$ between two vectors $\mathbf{v}_1 \equiv (x_1, y_1) = |\, r_1, h_1\rangle$ and $\mathbf{v}_2 \equiv (x_2, y_2) = |\, r_2, h_2\rangle$ by

$$h(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{h_2 - h_1}{1 + h_1 h_2} = h_2 \oplus (-h_1)$$

and use the pictorial conventions as shown in Figure 2 for an oriented triangle $\overrightarrow{A_1 A_2 A_3}$.

The *Relative half-turn formula* gives $h(\mathbf{v}_1, \mathbf{v}_2)$ in terms of the Cartesian coordinates x_1, y_1, x_2, y_2 and lengths $r_1 \equiv |\mathbf{v}_1|$ and $r_2 \equiv |\mathbf{v}_2|$:

$$h(\mathbf{v}_1, \mathbf{v}_2) = \frac{y_1 (r_2 - x_2) - y_2 (r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}.$$

The functions

$$C(h) \equiv \frac{1 - h^2}{1 + h^2} \qquad \text{and} \qquad S(h) \equiv \frac{2h}{1 + h^2}$$

together with the two closely related functions

$$T(h) \equiv \frac{2h}{1 - h^2} \qquad \text{and} \qquad M(h) \equiv \frac{2}{1 + h^2}$$

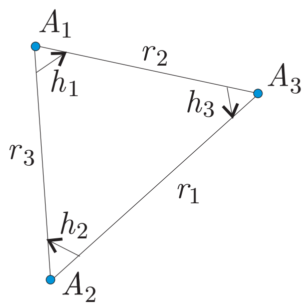

Figure 2: Lengths and relative half-turns of an oriented triangle $\overrightarrow{A_1A_2A_3}$

satisfy relations analogous to those of the usual circular functions, and may be used for integrating rational functions in a familiar way.

If $h_1 \oplus h_2 \equiv h_3$ and $C_1 \equiv C(h_1)$, $C_2 \equiv C(h_2)$ and $C_3 \equiv C(h_3)$, then the *Triple C formula* states that

$$C_1^2 + C_2^2 + C_3^2 = 1 + 2C_1C_2C_3.$$

There are rotor analogs for most of the usual trigonometric laws. For example, in an oriented triangle $\overrightarrow{A_1A_2A_3}$ with side lengths r_1, r_2 and r_3, and corresponding half-turns

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right) \qquad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \qquad \text{and} \qquad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$$

the Cosine law is replaced by the *Cross law in rotor form*:

$$r_3^2 = r_1^2 + r_2^2 - 2r_1r_2C(h_3).$$

From this

$$h_3^2 = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}.$$

The Sine law is replaced by the *Spread law in rotor form*:

$$\frac{S(h_1)}{r_1} = \frac{S(h_2)}{r_2} = \frac{S(h_3)}{r_3} = \frac{\sqrt{\mathcal{A}}}{2r_1r_2r_3}$$

where

$$\mathcal{A} \equiv (r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)$$

is the *quadrea* of the triangle. The fact that the angles sum to π, or $180°$, is replaced by the *Triangle turn formula*:

$$h_1h_2 + h_1h_3 + h_2h_3 = 1$$

which gives a linear equation for any one of the half-turns in terms of the other two.

For an oriented quadrilateral $\overrightarrow{A_1A_2A_3A_4}$ the relation between the half-turns

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_4}\right) \qquad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \qquad h_3 \equiv h\left(\overrightarrow{A_3A_4}, \overrightarrow{A_3A_2}\right) \qquad h_4 \equiv h\left(\overrightarrow{A_4A_4}, \overrightarrow{A_4A_3}\right)$$

is given by the *Quadrilateral turn formula:*

$$h_1 + h_2 + h_3 + h_4 = h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4.$$

Our main application is to the kinematics of Kepler-Newton orbits. We first discuss general particle motion in rotor coordinates, deriving some basic formulas, including the *Law of Conservation of Momentum* in this framework. Then we show that the *motion of a particle in a central inverse square force field defines a conic.* This historically important result is now within reach of a freshman calculus course.

Our treatment of the special parabolic case reveals that the half-turn h has natural physical significance as (in suitable coordinates) the directrix coordinate of the particle P moving parabolically around the focus O. The circle of velocities also relates naturally to the beautiful geometry of a parabola, which could happily play a bigger role in mathematics education.

2 The unit circle and the Cayley transform

Polar coordinates arise from the transcendental parametrization of the unit circle c_U with equation $x^2+y^2=1$ given by $\varphi(\theta) \equiv (\cos\theta, \sin\theta)$. In practice this generates vectors which are only *approximately* of unit length. There is a much older, and more exact, *rational parametrization:*

$$\mathbf{e}(h) \equiv (C(h), S(h)) \tag{3}$$

where

$$C(h) \equiv \frac{1-h^2}{1+h^2} \qquad \text{and} \qquad S(h) \equiv \frac{2h}{1+h^2}.$$

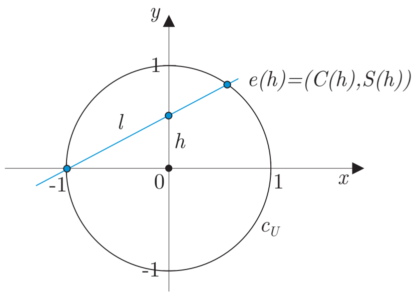

Figure 3: Rational parametrization of the unit circle

Geometrically $\mathbf{e}(h)$ is the point where the line l through $(-1,0)$ and $(0,h)$ meets c_U. If h is rational, then l will have rational coordinates, and since one of its meets with c_U is rational, the other will be also. The converse also holds; any rational point on c_U is of the form $\mathbf{e}(h)$, provided we also allow h to take on the *extended value* ∞, so that $e(\infty) = (-1,0)$. Other common examples are $e(0) = (1,0)$, $e(1) = (0,1)$ and $e(-1) = (0,-1)$.

The rational parametrization has a modern formulation in terms of linear algebra. If X is a skew-symmetric matrix for which $I+X$ is invertible, then the **Cayley transform** of X is defined to be the orthogonal matrix

$$c(X) \equiv \frac{I-X}{I+X}.$$

In the 2×2 case, if

$$X = \begin{pmatrix} 0 & -h \\ h & 0 \end{pmatrix} \qquad \text{then} \qquad c(X) = \begin{pmatrix} C(h) & S(h) \\ -S(h) & C(h) \end{pmatrix} \equiv \sigma_h. \tag{4}$$

If we also define

$$\sigma_\infty \equiv \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{5}$$

which is consistent (in a limiting sense) with $h=\infty$, then the orthogonal matrices σ_h for h an extended rational number (that is, including the value ∞) bijectively represent rational rotations. This gives us an *algebraic* alternative to the usual exponential map between the line and the group of rotations.

3 Rotor coordinates

If h is a rational number, then $\mathbf{e}(h) \equiv (C(h), S(h))$ is a rational vector of unit length. For any rational number $r>0$, the vector $\mathbf{v} = r\mathbf{e}(h)$ is then also a rational vector, and the usual Cartesian coordinates for $\mathbf{v}$ are

$$x = r\left(\frac{1-h^2}{1+h^2}\right) \qquad \text{and} \qquad y = r\left(\frac{2h}{1+h^2}\right). \tag{6}$$

The number

$$r = r(\mathbf{v}) \equiv \sqrt{x^2+y^2} \tag{7}$$

is the **length** of $\mathbf{v}$, and

$$h = h(\mathbf{v})$$

is the **half-turn** of $\mathbf{v}$. In the special case of $\mathbf{w} \equiv (0, -1)$, we define $h(r\mathbf{w}) = \infty$ for any $r > 0$.

The quantities r and h determine $\mathbf{v}$, and will be called **rotor coordinates** for $\mathbf{v}$, written

$$\mathbf{v} = |\, r, h\rangle .$$

The above formulas extend to more general vectors $\mathbf{v} = (x, y)$, but in this case r will typically exist in a *quadratic extension* of the field containing x and y, which also contains h because of the following important result. We give two proofs.

Theorem 1 (Half-turn formula) *If* $\mathbf{v} \equiv (x, y)$ *has length* $r \equiv \sqrt{x^2 + y^2}$ *and* $y \neq 0$, *then*

$$h(\mathbf{v}) = \frac{r - x}{y}. \tag{8}$$

Proof. To find $h \equiv h(\mathbf{v})$, normalize to obtain the unit vector

$$\frac{\mathbf{v}}{r} = \left(\frac{x}{r}, \frac{y}{r}\right)$$

which is collinear with the vectors $(-1, 0)$ and $(0, h)$ as in Figure 1. It follows from similar triangles that

$$\frac{h}{1} = \frac{y/r}{1 + x/r} = \frac{y}{r + x} = \frac{r - x}{y},$$

the last equality since $y^2 = r^2 - x^2$.

Alternatively, use (6) to see that

$$\frac{r - x}{y} = \frac{1 + h^2}{2h} - \frac{1 - h^2}{2h} = h. \quad \blacksquare$$

In the special case when $y = 0$, the half-turn h is either 0 or ∞, depending on whether x is positive or negative. In a diagram we represent the half-turn h of a vector $\mathbf{v}$ as shown in Figure 1.

4 Examples of half-turns for unit vectors

Rotor coordinates describe rational vectors in the plane without prior set-up of the full real number system. They provide a useful, simpler and often more powerful alternative to polar coordinates.

The table below gives some examples of unit vectors $\mathbf{v} = (x, y)$, so that $r = 1$, together with their half-turns $h \equiv h(\mathbf{v})$, and their corresponding angles θ, where we write $\theta \approx h$. We restrict to the cases for which h is positive, so that $y \geq 0$, with corresponding angles θ in the range $0 \leq \theta \leq 180°$. If we negate a half-turn, then the corresponding angle is also negated.

Unit vector v		**Half - turn** h		**Angle** θ
$\left(1/\sqrt{2}, 1/\sqrt{2}\right)$		$\frac{1-1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} - 1$		45°
$\left(-1/\sqrt{2}, 1/\sqrt{2}\right)$		$\frac{1+1/\sqrt{2}}{1/\sqrt{2}} = \sqrt{2} + 1$		135°
$\left(\sqrt{3}/2, 1/2\right)$		$\frac{1-\sqrt{3}/2}{1/2} = 2 - \sqrt{3}$		30°
$\left(-\sqrt{3}/2, 1/2\right)$		$\frac{1+\sqrt{3}/2}{1/2} = 2 + \sqrt{3}$		150°
$\left(1/2, \sqrt{3}/2\right)$		$\frac{1-1/2}{\sqrt{3}/2} = 1/\sqrt{3}$		60°
$\left(-1/2, \sqrt{3}/2\right)$		$\frac{1+1/2}{\sqrt{3}/2} = \sqrt{3}$		120°
$\left(\frac{\sqrt{5}-1}{4}, \frac{\sqrt{10+2\sqrt{5}}}{4}\right)$		$\sqrt{5 - 2\sqrt{5}}$		72°
$\left(\frac{-\sqrt{5}-1}{4}, \frac{\sqrt{10-2\sqrt{5}}}{4}\right)$		$\sqrt{5 + 2\sqrt{5}}$		144°

5 Projective formulation and the circle sum

While the half-turn h is very convenient for applications, having to treat the special case $h = \infty$ separately becomes an inconvenience for theoretical work. This may be overcome by moving to the more natural *projective parametrization* of the unit circle, which we now explain.

The *projective line* over the rationals consists of proportions

$$\alpha \equiv [t : u]$$

where t and u are rational numbers, not both zero. By scaling these may be taken to be integers.

The rational half-turn $h = h(\mathbf{v}) = t/u$ of a vector $\mathbf{v}$ corresponds to the *projective half-turn*

$$\alpha(\mathbf{v}) = [h : 1] = [t : u]$$

while the extended rational half-turn $h = h(\mathbf{w}) = \infty$ of the vector $\mathbf{w} \equiv (-1, 0)$ corresponds to the *projective half-turn*

$$\alpha(\mathbf{w}) = [1 : 0] .$$

In this way both cases can be dealt with uniformly. The bijection between projective half-turns and the unit circle is

$$\mathbf{e}([t : u]) \equiv \left[\frac{u^2 - t^2}{u^2 + t^2}, \frac{2ut}{u^2 + t^2}\right].$$

In parallel with (4), for a proportion $\alpha \equiv [t : u]$ define the **rotation matrix**

$$\sigma_\alpha \equiv \frac{1}{(u^2 + t^2)} \begin{pmatrix} u^2 - t^2 & 2ut \\ -2ut & u^2 - t^2 \end{pmatrix}$$

acting on a (row) vector $\mathbf{v} = (x, y)$ on the right by $v \to v\sigma_\alpha$. Here is a key theorem.

Theorem 2 (Circle sum) *If $\alpha_1 \equiv [t_1 : u_1]$ and $\alpha_2 \equiv [t_2 : u_2]$ then*

$$\sigma_{\alpha_1}\sigma_{\alpha_2} = \sigma_\alpha$$

where

$$\alpha \equiv [u_1 t_2 + u_2 t_1 : u_1 u_2 - t_1 t_2] \equiv \alpha_1 \oplus \alpha_2$$

defines the ***circle sum*** *of the two proportions α_1 and α_2.*

Proof. Note first that the circle sum $\alpha \equiv \alpha_1 \oplus \alpha_2$ is well-defined, in that if we scale the entries in either α_1 or α_2, the proportion α is unchanged, and because Fibonacci's identity

$$(u_1 t_2 + u_2 t_1)^2 + (t_1 t_2 - u_1 u_2)^2 = \left(t_1^2 + u_1^2\right)\left(t_2^2 + u_2^2\right) \tag{9}$$

ensures that the entries of α are not both zero. The latter also ensures that we need only check that

$$\begin{pmatrix} u_1^2 - t_1^2 & 2u_1 t_1 \\ -2u_1 t_1 & u_1^2 - t_1^2 \end{pmatrix} \begin{pmatrix} u_2^2 - t_2^2 & 2u_2 t_2 \\ -2u_2 t_2 & u_2^2 - t_2^2 \end{pmatrix} = \begin{pmatrix} (u_1 u_2 - t_1 t_2)^2 - (u_1 t_2 + u_2 t_1)^2 & 2(u_1 u_2 - t_1 t_2)(u_1 t_2 + u_2 t_1) \\ -2(u_1 u_2 - t_1 t_2)(u_1 t_2 + u_2 t_1) & (u_1 u_2 - t_1 t_2)^2 - (u_1 t_2 + u_2 t_1)^2 \end{pmatrix}.$$

This in turns rests on the identities

$$\left(u_1^2 - t_1^2\right)\left(u_2^2 - t_2^2\right) - (2u_1 t_1)(2u_2 t_2) = (u_1 u_2 - t_1 t_2)^2 - (u_1 t_2 + u_2 t_1)^2 \tag{10}$$

$$\left(u_1^2 - t_1^2\right)(2u_2 t_2) + (2u_1 t_1)\left(u_2^2 - t_2^2\right) = 2(u_1 u_2 - t_1 t_2)(u_1 t_2 + u_2 t_1). \quad \blacksquare \tag{11}$$

The circle sum is associative (since it corresponds, by the theorem, to matrix multiplication), commutative, and has identity $[0 : 1]$. The inverse of $[t : u]$ is $[-t : u]$. The map $\alpha \to \sigma_\alpha$ defines a homomorphism between the group of projective half-turns under circle sum, and the multiplicative group of rational rotation matrices.

6 Rational circle sums and turn polynomials

When we restate the Circle sum theorem in terms of rational half-turns h, we find that

$$\sigma_{h_1}\sigma_{h_2}=\sigma_h$$

where

$$h=\frac{h_1+h_2}{1-h_1h_2}\equiv h_1\oplus h_2. \tag{12}$$

This **rational circle sum** extends to values of ∞ by limiting arguments, or by going back to the projective formulation. The identity is $h=0$, and the inverse of h is $-h$, so that

$$(-h_1)\oplus(-h_2)=-(h_1\oplus h_2). \tag{13}$$

Example 1 *The half-turn that corresponds to an angle of* $45°+30°=75°$ *is*

$$\begin{aligned} h &= \left(\sqrt{2}-1\right)\oplus\left(2-\sqrt{3}\right)=\frac{(\sqrt{2}-1)+(2-\sqrt{3})}{1-(\sqrt{2}-1)(2-\sqrt{3})} \\ &= \sqrt{3}+\sqrt{6}-\sqrt{2}-2. \ \diamond \end{aligned}$$

The circle sum operation is commutative and also associative, so that

$$(h_1\oplus h_2)\oplus h_3=h_1\oplus(h_2\oplus h_3)=\frac{h_1+h_2+h_3-h_1h_2h_3}{1-(h_1h_2+h_2h_3+h_1h_3)} \tag{14}$$

and similarly

$$h_1\oplus h_2\oplus h_3\oplus h_4=\frac{h_1+h_2+h_3+h_4-(h_1h_2h_3+h_1h_2h_4+h_1h_3h_4+h_2h_3h_4)}{1-(h_1h_3+h_1h_4+h_2h_3+h_2h_4+h_3h_4+h_1h_2)+h_1h_2h_3h_4}. \tag{15}$$

Theorem 3 (Multiple circle sums) *For any natural number n, any rational numbers $h_1,h_2,\cdots,h_n$, and any natural number k in the range $1\leq k\leq n$, let*

$$s_k\equiv s_k(h_1,h_2,\cdots,h_n)\equiv\sum_{\{i_1,i_2,\cdots,i_k\}\subseteq\{1,2,\cdots,n\}}h_{i_1}h_{i_2}\cdots h_{i_k}.$$

If $n=2m$ is even then

$$h_1\oplus h_2\oplus\cdots\oplus h_n=\frac{s_1-s_3+\cdots+(-1)^{m-1}s_{2m-1}}{1-s_2+s_4-\cdots+(-1)^m s_{2m}}$$

while if $n=2m+1$ is odd then

$$h_1\oplus h_2\oplus\cdots\oplus h_n=\frac{s_1-s_3+\cdots+(-1)^{m}s_{2m+1}}{1-s_2+s_4-\cdots+(-1)^m s_{2m}}.$$

Proof. We proceed by induction. We may check that for $n=1$ and $n=2$ the formulas are correct. Assume they are true for n, and now to prove the corresponding formula for $n+1$, for k in the range $1\leq k\leq n+1$ set

$$\mu_k\equiv\mu_k(h_1,h_2,\cdots,h_n,h_{n+1})\equiv\sum_{\{i_1,i_2,\cdots,i_k\}\subseteq\{1,2,\cdots,n+1\}}h_{i_1}h_{i_2}\cdots h_{i_k}.$$

If $n=2m$ then

$$(h_1\oplus h_2\oplus\cdots\oplus h_n)\oplus h_{n+1}=\frac{\left(\frac{s_1-s_3+\cdots+(-1)^{m-1}s_{2m-1}}{1-s_2+\cdots+(-1)^m s_{2m}}\right)+h_{n+1}}{1-\left(\frac{s_1-s_3+\cdots+(-1)^{m-1}s_{2m-1}}{1-s_2+\cdots+(-1)^m s_{2m}}\right)\times h_{n+1}}$$

while if $n=2m+1$ then

$$(h_1\oplus h_2\oplus\cdots\oplus h_n)\oplus h_{n+1}=\frac{\left(\frac{s_1-s_3+\cdots+(-1)^{m}s_{2m+1}}{1-s_2+s_4-\cdots+(-1)s_{2m}}\right)+h_{n+1}}{1-\left(\frac{s_1-s_3+\cdots+(-1)^{m}s_{2m+1}}{1-s_2+s_4-\cdots+(-1)s_{2m}}\right)\times h_{n+1}}.$$

The induction then rests on two identities, the first when $n = 2m$ being

$$\begin{aligned} & \left(s_1 - s_3 + \cdots + (-1)^{m-1} s_{2m-1}\right) + \left(1 - s_2 + \cdots + (-1)^m s_{2m}\right) h_{n+1} \\ = \; & \mu_1 - \mu_3 + \cdots + (-1)^m \mu_{2m+1} \end{aligned}$$

and the second when $n = 2m + 1$ being

$$\begin{aligned} & \left(s_1 - s_3 + \cdots + (-1)^m s_{2m+1}\right) + \left(1 - s_2 + s_4 - \cdots + (-1) s_{2m}\right) h_{n+1} \\ = \; & \mu_1 - \mu_3 + \cdots + (-1)^m \mu_{2m+1}. \ \blacksquare \end{aligned}$$

Taking the circle sum of h with itself yields a rational function of h which we call $U_2(h)$, namely

$$h \oplus h = \frac{2h}{1 - h^2} \equiv U_2(h).$$

Continuing, we get a sequence $U_n(h)$ of rational functions, which we call the **turn functions**:

$$\begin{aligned} h \oplus h \oplus h &= \frac{3h - h^3}{1 - 3h^2} \equiv U_3(h) \\ h \oplus h \oplus h \oplus h &= \frac{4h - 4h^3}{1 - 6h^2 + h^4} \equiv U_4(h) \\ h \oplus h \oplus h \oplus h \oplus h &= \frac{5u - 10u^3 + u^5}{1 - 10u^2 + 5u^4} \equiv U_5(h). \end{aligned}$$

The pattern of binomial coefficients follows directly from the Multiple circle sums theorem. These functions have been known for centuries (see [3, pg 155]), although our name for them is new. They warrant more study.

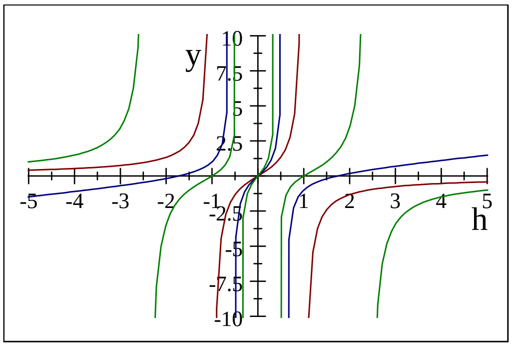

Figure 4: Turn functions U_2(red), U_3 (blue), and U_4 (green)

Example 2 *If we wish to bisect the sector created by two vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *with* $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$*, then we need find a half-turn* k *satisfying*

$$U_2(k) \equiv k \oplus k = \frac{2k}{1 - k^2} = h.$$

This quadratic equation $hk^2 + 2k - h = 0$ *has discriminant* $4\left(1 + h^2\right)$*, so that we require* $1 + h^2$ *to be a square.* ◇

Example 3 *If we wish to trisect the sector created by two vectors* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *with* $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$*, then we need find a half-turn* k *satisfying*

$$U_3(k) \equiv k \oplus k \oplus k = \frac{3k - k^3}{1 - 3k^2} = h.$$

This yields the cubic equation $k^3 - 3hk^2 - 3k + h = 0$ *which we may transform in the usual way by setting* $k = y + h$ *to get*

$$y^3 = py + q$$

where $p = 3\left(1 + h^2\right)$ *and* $q = 2h\left(1 + h^2\right)$*. The discriminant* $4p^2 - 27q^2$ *is* $108\left(1 + h^2\right)^2$*.* ◇

Example 4 *Suppose we want to verify the half-turns h associated to the fifth roots of unity. It means solving $U_5(h) = 0$, namely*

$$5h - 10h^3 + h^5 = h\left(h^4 - 10h^2 + 5\right) = 0.$$

Besides the obvious solution $h = 0$, we also get $h = \pm\sqrt{5 - 2\sqrt{5}}, \pm\sqrt{5 + 2\sqrt{5}}$, as in our earlier table. $\diamond$

7 Half-turn transformations

Since this paper is oriented to applications, we will stick with the view of half-turns as extended rational numbers h, and refer to (12) as simply the circle sum. The reader should have little difficulty in formulating projective versions if required.

Theorem 4 (Half-turn transformations) *Suppose that the vector $\mathbf{v}$ has half-turn h. Then the reflection of $\mathbf{v}$ in the x-axis has half-turn $-h$, the reflection of $\mathbf{v}$ in the y-axis has half turn h^{-1}, the vector $-\mathbf{v}$ has half-turn $-h^{-1}$, while the reflection of $\mathbf{v}$ in the line $y = x$ and the rotation of $\mathbf{v}$ by a one-quarter of the full circle in the positive direction have respective half-turns*

$$\frac{1-h}{1+h} \qquad \text{and} \qquad \frac{1+h}{1-h}.$$

Proof. These are easy calculations, such as

$$1 \oplus (-h) = \frac{1-h}{1+h} \qquad \text{and} \qquad 1 \oplus h = \frac{1+h}{1-h}. \quad \blacksquare$$

The theorem can also be used to relate angle transformations to half-turn transformations. Denote $\theta \approx h$ the relation between an angle and a half-turn as before. Then $-\theta \approx -h$ and

$$\begin{aligned} 180^\circ - \theta &\approx \frac{1}{h} \qquad \text{and} \qquad 180^\circ + \theta \approx -\frac{1}{h} \\ 90^\circ - \theta &\approx \frac{1-h}{1+h} \qquad \text{and} \qquad 90^\circ + \theta \approx \frac{1+h}{1+h}. \end{aligned}$$

Figure 5 shows the effect of reflections in the coordinate axes and the lines $y = \pm x$ on the half-turn h.

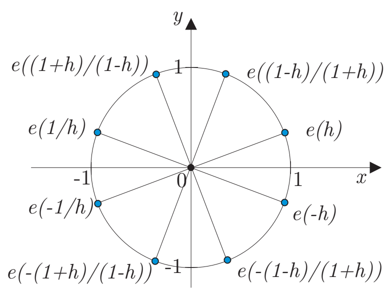

Figure 5: Reflections and half-turns

One can also easily check that

$$\frac{1}{h_1} \oplus \frac{1}{h_2} = -\left(h_1 \oplus h_2\right).$$

Example 5 *Many unit vectors, of interest already to the Pythagoreans, have corresponding angles which do not have tidy values in the radian or degree systems, and so are seldom used in high school examples or tests, despite their simplicity and attractiveness. For example the vector $(3/5, 4/5)$ has half-turn $h = 1/2$, the vector $(4/5, 3/5)$ has $h = 1/3$, the vector $(5/13, 12/13)$ has $h = 2/3$ and the vector $(12/13, 5/13)$ has $h = 1/5$.* $\diamond$

Example 6 *To find the product of the rotations σ_α corresponding to the unit vectors $(3/5, 4/5)$ and $(5/13, 12/13)$, compute*

$$\frac{1}{2} \oplus \frac{2}{3} = \frac{\frac{1}{2} + \frac{2}{3}}{1 - \frac{1}{2} \times \frac{2}{3}} = \frac{7}{4}.$$

so that

$$\sigma_{1/2}\sigma_{2/3} = \sigma_{7/4} = c\left(\begin{pmatrix} 0 & -7/4 \\ 7/4 & 0 \end{pmatrix}\right) = \frac{1}{65}\begin{pmatrix} -33 & 56 \\ -56 & -33 \end{pmatrix}. \quad \diamond$$

Example 7 *Here are a few rotor forms for non-unit vectors. If $\mathbf{v} \equiv (1, 2)$ then $r = \sqrt{5}$ and*

$$h = \frac{\sqrt{5} - 1}{2} \approx 0.618\,03$$

the Golden ratio. If $\mathbf{v} \equiv (2, 1)$ then

$$h = \sqrt{5} - 2 \approx 0.236\,07.$$

If $\mathbf{v} \equiv (1, 3)$ then $r = \sqrt{10}$ and

$$h = \frac{\sqrt{10} - 1}{3} \approx 0.720\,76.$$

Clearly once we have found the length r, (8) makes it easy to compute the half-turn h. $\diamond$

8 The rational functions C, S, T and M

The functions

$$C(h) \equiv \frac{1 - h^2}{1 + h^2} \qquad \text{and} \qquad S(h) \equiv \frac{2h}{1 + h^2}$$

are important enough to have names; we call them the **capital** C and **capital** S functions respectively. The closely related **capital** T **function** is defined as

$$T(h) \equiv \frac{S(h)}{C(h)} = \frac{2h}{1 - h^2}.$$

These three functions have graphs, over the rational numbers, as shown in Figure 6. They satisfy analogs of

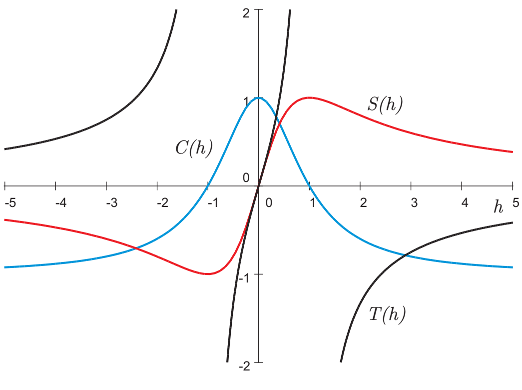

Figure 6: Graphs of $C(h)$, $S(h)$ and $T(h)$

well-known properties of the transcendental circular functions $\cos\theta$, $\sin\theta$ and $\tan\theta$. The most obvious such relations are

$$C(h)^2 + S(h)^2 = 1 \tag{16}$$

together with the symmetry conditions

$$C(-h) = C(h) \qquad S(-h) = -S(h) \qquad \text{and} \qquad T(-h) = -T(h).$$

The addition formulas for C and S are

$$C(h_1 \oplus h_2) = C(h_1)C(h_2) - S(h_1)S(h_2) \tag{17}$$

$$S(h_1 \oplus h_2) = C(h_1)S(h_2) + C(h_2)S(h_1) \tag{18}$$

which are essentially contained in the identities (10) and (11).

The addition formula for T relates directly to the circle sum:

$$T(h_1 \oplus h_2) = \frac{T(h_1) + T(h_2)}{1 - T(h_1)T(h_2)} = T(h_1) \oplus T(h_2)$$

and is a consequence of the identity

$$\frac{2\left(\frac{h_1+h_2}{1-h_1h_2}\right)}{1-\left(\frac{h_1+h_2}{1-h_1h_2}\right)^2} = \frac{\left(\frac{2h_1}{1-h_1^2}\right)+\left(\frac{2h_2}{1-h_2^2}\right)}{1-\left(\frac{2h_1}{1-h_1^2}\right)\left(\frac{2h_2}{1-h_2^2}\right)}.$$

Another important function is the **capital** M **function**

$$M(h) \equiv \frac{2}{1+h^2} = 1 + C(h) = \frac{S(h)}{h}$$

whose main significance will become clear in (23), but is already involved in the following.

Theorem 5 (C and S derivative) *The derivatives of C and S are*

$$\frac{dC}{dh}(h) = -S(h)\ M(h) \qquad \text{and} \qquad \frac{dS}{dh}(h) = C(h)\ M(h).$$

Proof. This is a first-year calculus computation. ■

Theorem 6 (C and S second order derivative) *Both $C(h)$ and $S(h)$ satisfy the second order differential equation*

$$\frac{1}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{df}{dh}\right) + f = 0.$$

Proof. This follows by combining both formulas of the previous theorem. ■

9 Relative half-turns between vectors

Up to now we have defined the half-turn of a single vector, which depends on the choice of positive x-axis. We now define the **half-turn between two vectors** $\mathbf{v}_1 = |\,r_1, h_1\rangle$ and $\mathbf{v}_2 = |\,r_2, h_2\rangle$, or their **relative half-turn**, to be

$$h = h(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{h_2 - h_1}{1 + h_1h_2} = h_2 \oplus (-h_1).$$

It follows that

$$h_1 \oplus h = h_2.$$

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are in opposite directions, then $h_1h_2 = -1$, so that $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$ is interpreted as having the value ∞.

The relative half-turn is an oriented quantity, in that

$$h(\mathbf{v}_2, \mathbf{v}_1) = -h(\mathbf{v}_1, \mathbf{v}_2).$$

Example 8 *If $\mathbf{v}_1 \equiv (3,2)$ and $\mathbf{v}_2 = (2,5)$ then*

$$h(\mathbf{v}_1, \mathbf{v}_2) = h_2 \oplus (-h_1) = \frac{\left(\frac{\sqrt{29}-2}{5}\right) - \left(\frac{\sqrt{13}-3}{2}\right)}{1 + \left(\frac{\sqrt{29}-2}{5}\right)\left(\frac{\sqrt{13}-3}{2}\right)} = \frac{1}{11}\sqrt{377} - \frac{16}{11}. \ \diamond$$

If we want an *undirected* quantity between $\mathbf{v}_1$ and $\mathbf{v}_2$, we may take the square $H \equiv h^2$ of the half-turn $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$. Note that the spread s between $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$s = \frac{4h^2}{(1+h^2)^2} = \frac{4H}{(1+H)^2}.$$

While the half-turn between vectors is unchanged if either is multiplied by a positive number, this is no longer true if we multiply by -1.

Example 9 *For any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$,*

$$h(-\mathbf{v}_1, \mathbf{v}_2) = -\frac{1}{h(\mathbf{v}_1, \mathbf{v}_2)}.$$

This follows from the half-turn transformation theorem; for if $h_1 \equiv h(\mathbf{v}_1)$ and $h_2 \equiv h(-\mathbf{v}_1)$ then

$$h(-\mathbf{v}_1) = \frac{1}{h_1},$$

so that

$$h(-\mathbf{v}_1, \mathbf{v}_2) = \frac{h_2 - (-1/h_1)}{1 + (-1/h_1)h_2} = \frac{1 + h_1 h_2}{h_1 - h_2} = -\frac{1}{h(\mathbf{v}_1, \mathbf{v}_2)}. \quad \diamond$$

Example 10 *Applying the previous example twice we see that for any vectors $\mathbf{v}_1$ and $\mathbf{v}_2$,*

$$h(-\mathbf{v}_1, -\mathbf{v}_2) = h(\mathbf{v}_1, \mathbf{v}_2). \quad \diamond$$

Theorem 7 (Relative half-turn formula) *If $\mathbf{v}_1 \equiv (x_1, y_1)$ and $\mathbf{v}_2 \equiv (x_2, y_2)$ with $r_1 \equiv r(\mathbf{v}_1)$ and $r_2 \equiv r(\mathbf{v}_2)$, then*

$$h = h(\mathbf{v}_1, \mathbf{v}_2) = \frac{y_1(r_2 - x_2) - y_2(r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}.$$

Proof. From the Half-turn formula

$$h_1 \equiv h(\mathbf{v}_1) = \frac{r_1 - x_1}{y_1} \qquad \text{and} \qquad h_2 \equiv h(\mathbf{v}_2) = \frac{r_2 - x_2}{y_2},$$

so that

$$\begin{aligned} h &= h(\mathbf{v}_1, \mathbf{v}_2) \equiv \frac{h_2 - h_1}{1 + h_1 h_2} = \frac{\left(\frac{r_2 - x_2}{y_2}\right) - \left(\frac{r_1 - x_1}{y_1}\right)}{1 + \left(\frac{r_1 - x_1}{y_1}\right)\left(\frac{r_2 - x_2}{y_2}\right)} \\ &= \frac{y_1(r_2 - x_2) - y_2(r_1 - x_1)}{y_1 y_2 + (r_1 - x_1)(r_2 - x_2)}. \quad \blacksquare \end{aligned}$$

The next result shows that the relative half-turn is invariant under the rotations σ_h introduced in (4) and (5).

Theorem 8 (Half-turn invariance) *For vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and any half turn h*

$$h(\mathbf{v}_1, \mathbf{v}_2) = h(\mathbf{v}_1 \sigma_h, \mathbf{v}_2 \sigma_h).$$

Proof. If $h_1 \equiv h(\mathbf{v}_1)$ and $h_2 \equiv h(\mathbf{v}_2)$ then

$$h(\mathbf{v}_1 \sigma_h) = h_1 \oplus h \qquad \text{and} \qquad h(\mathbf{v}_2 \sigma_h) = h_2 \oplus h.$$

Now use (13), and the group properties of the circle sum to get

$$\begin{aligned} h(\mathbf{v}_1 \sigma_h, \mathbf{v}_2 \sigma_h) &= (h_2 \oplus h) \oplus (-(h_1 \oplus h)) \\ &= h_2 \oplus h \oplus (-h_1) \oplus (-h) \\ &= h_2 \oplus (-h_1) \oplus h \oplus (-h) \\ &= h_2 \oplus (-h_1) \oplus 0 = h_2 \oplus (-h_1) \\ &= h(\mathbf{v}_1, \mathbf{v}_2). \quad \blacksquare \end{aligned}$$

Theorem 9 (Triple C formula) *If $h_1 \oplus h_2 \equiv h_3$ and $C_1 \equiv C(h_1)$, $C_2 \equiv C(h_2)$ and $C_3 \equiv C(h_3)$, then*

$$C_1^2 + C_2^2 + C_3^2 = 1 + 2C_1C_2C_3.$$

Proof. Combine (17) with (16) to obtain

$$(C_3 - C_1C_2)^2 = \left(1 - C_1^2\right)\left(1 - C_2^2\right).$$

Now expand to get the result. ■

There is no such simple relation between the three values $S_1 \equiv S(h_1)$, $S_2 \equiv S(h_2)$ and $S_3 \equiv S(h_3)$. However their squares, the spreads $s_1 \equiv S_1^2$, $s_2 \equiv S_2^2$ and $s_3 \equiv S_3^2$, satisfy the *Triple spread formula*

$$(s_1 + s_2 + s_3)^2 = 2\left(s_1^2 + s_2^2 + s_3^2\right) + 4s_1s_2s_3, \tag{19}$$

one of the main laws of rational trigonometry. This can be derived directly from the Triple C formula by rewriting and squaring it to obtain

$$(2 - (s_1 + s_2 + s_3))^2 = 4(1 - s_1)(1 - s_2)(1 - s_3),$$

and then rearranging.

Theorem 10 (Three half-turns) *If $\mathbf{v}_1, \mathbf{v}_2$ and $\mathbf{v}_3$ are three vectors with $h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, $h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3)$ and $h_{13} \equiv h(\mathbf{v}_1, \mathbf{v}_3)$ then*

$$h_{13} = h_{12} \oplus h_{23}.$$

Proof. If $h_1 \equiv h(\mathbf{v}_1)$, $h_2 \equiv h(\mathbf{v}_2)$ and $h_3 \equiv h(\mathbf{v}_3)$, then

$$\begin{aligned} h_{12} \oplus h_{23} &= (h_2 \oplus (-h_1)) \oplus (h_3 \oplus (-h_2)) \\ &= h_3 \oplus h_2 \oplus (-h_2) \oplus (-h_1) \\ &= h_3 \oplus (-h_1) = h_{13}. \ ■ \end{aligned}$$

10 The Cross law and vector trigonometry

In this section we establish formulas of vector trigonometry relating to an oriented triangle $\overrightarrow{A_1A_2A_3}$ with respective side lengths r_1, r_2 and r_3, and relative half-turns $h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right)$, $h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right)$ and $h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$.

Throughout we work in the realm of extended rational numbers and quadratic extensions. We will state the main result in terms of two vectors and the half-turn between them.

Theorem 11 (Cross law–rotor form) *If vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ have respective lengths r_1 and r_2, and half-turn $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, then $\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1$ has length r_3, where*

$$r_3^2 = r_1^2 + r_2^2 - 2r_1r_2C(h).$$

Proof. Suppose that $\mathbf{v}_1 \equiv |\, r_1, h_1\rangle$ and $\mathbf{v}_2 \equiv |\, r_1, h_1\rangle$ so that

$$\mathbf{v}_1 = (r_1C(h_1), r_1S(h_1)) \qquad \text{and} \qquad \mathbf{v}_2 = (r_2C(h_2), r_2S(h_2))$$

and

$$h \equiv h(\mathbf{v}_1, \mathbf{v}_2) = \frac{h_2 - h_1}{1 + h_1h_2}.$$

Then

$$\mathbf{v}_3 = \mathbf{v}_2 - \mathbf{v}_1 = (r_2C(h_2) - r_1C(h_1), r_2S(h_2) - r_1S(h_1)).$$

Now compute that

$$\begin{aligned} r_3^2 &= (r_2 C(h_2) - r_1 C(h_1))^2 + (r_2 S(h_2) - r_1 S(h_1))^2 \\ &= r_1^2 \left(C(h_1)^2 + S(h_1)^2 \right) + r_2^2 \left(C(h_2)^2 + S(h_2)^2 \right) - 2 r_1 r_2 \left(C(h_1) C(h_2) + S(h_1) S(h_2) \right) \\ &= r_1^2 + r_2^2 - 2 r_1 r_2 C(h_2 \oplus (-h_1)) = r_1^2 + r_2^2 - 2 r_1 r_2 C(h) \end{aligned}$$

where we have used (16) and the addition formula (17) for $C(h)$. ■

Recall that the *triangle inequalities* for a triangle with side lengths r_1, r_2 and r_3 are

$$(r_1 - r_2)^2 \leq r_3^2 \leq (r_1 + r_2)^2.$$

So r_3^2 is a convex combination of $(r_1 - r_2)^2$ and $(r_1 + r_2)^2$, and the Cross law above makes this explicit, as it may be rewritten in the form

$$r_3^2 = \frac{1}{1+h^2} (r_1 - r_2)^2 + \frac{h^2}{1+h^2} (r_1 + r_2)^2 = \frac{1}{1+H} (r_1 - r_2)^2 + \frac{H}{1+H} (r_1 + r_2)^2$$

where $H \equiv h^2$. The next result provides an alternative to the Relative half-turn formula.

Theorem 12 (Vectors half-turn) *If $\mathbf{v}_1 \equiv (x_1, y_1)$ and $\mathbf{v}_2 \equiv (x_2, y_2)$ are vectors with respective lengths r_1 and r_2, and relative half-turn $h \equiv h(\mathbf{v}_1, \mathbf{v}_2)$, then*

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)} = \frac{\left(x_1^2 + y_1^2\right)\left(x_2^2 + y_2^2\right) - 2 r_1 r_2 (x_1 x_2 + y_1 y_2) + (x_1 x_2 + y_1 y_2)^2}{(x_1 y_2 - x_2 y_1)^2}.$$

Proof. Apply the Cross law to the triangle formed from the vectors $\mathbf{v}_1$ and $\mathbf{v}_2$, with side lengths r_1, r_2 and $r_3 \equiv (x_2 - x_1)^2 + (y_2 - y_1)^2$, to get

$$C(h) = \frac{1-h^2}{1+h^2} = \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2} = \frac{x_1 x_2 + y_1 y_2}{r_1 r_2}$$

and solve for h^2 to get

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)}.$$

Now multiply numerator and denominator by the numerator, and use Fibonacci's identity (9). ■

Example 11 *For $\mathbf{v}_1 \equiv (3, 2)$ and $\mathbf{v}_2 = (2, 5)$ we get*

$$h^2 = \frac{r_1 r_2 - (x_1 x_2 + y_1 y_2)}{r_1 r_2 + (x_1 x_2 + y_1 y_2)} = \frac{\sqrt{13}\sqrt{29} - 16}{\sqrt{13}\sqrt{29} + 16} = \frac{633}{121} - \frac{32}{121}\sqrt{377}.$$

Comparing with example 8, you may check that this is indeed $\left(\frac{1}{11}\sqrt{377} - \frac{16}{11}\right)^2$. ⋄

Theorem 13 (Triangle half-turn) *If an oriented triangle $\overrightarrow{A_1 A_2 A_3}$ has respective side lengths r_1, r_2 and r_3 and half-turn $h_3 \equiv h\left(\overrightarrow{A_3 A_1}, \overrightarrow{A_3 A_2}\right)$, then*

$$h_3^2 = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}. \tag{20}$$

Proof. We know from the Cross law that $r_3^2 = r_1^2 + r_2^2 - 2 r_1 r_2 C(h_3)$ so that

$$C(h_3) = \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}.$$

It follows that

$$h_3^2 = \frac{1 - \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}}{1 + \frac{r_1^2 + r_2^2 - r_3^2}{2 r_1 r_2}} = \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2}.$$

Now rewrite this as

$$h_3^2 = \frac{(r_1 - r_2 - r_3)(r_2 - r_1 - r_3)}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}. \quad \blacksquare \tag{21}$$

If the quadrances of the triangle $\overline{A_1A_2A_3}$ are denoted $Q_1 \equiv r_1^2$, $Q_2 \equiv r_2^2$, and $Q_3 \equiv r_3^2$, then by a rational version of Heron's formula, which we call *Archimedes' formula* (see [5, Theorem 29, page 70]), the *quadrea* of the triangle

$$\begin{aligned} \mathcal{A} &\equiv (Q_1 + Q_2 + Q_3)^2 - 2\left(Q_1^2 + Q_2^2 + Q_3^2\right) \\ &= (r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3) \end{aligned}$$

is 16 times the square of the triangle's area.

Theorem 14 (Sine law–rotor form) *If an oriented triangle $\overrightarrow{A_1A_2A_3}$ has respective side lengths r_1, r_2 and r_3, half-turns $h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right)$, $h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right)$ and $h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right)$, and quadrea $\mathcal{A}$, then*

$$\frac{S(h_1)}{r_1} = \frac{S(h_2)}{r_2} = \frac{S(h_3)}{r_3} = \frac{\sqrt{\mathcal{A}}}{2r_1r_2r_3}.$$

Proof. Given h_3^2 as in (20),

$$1 + h_3^2 = 1 + \frac{r_3^2 - (r_1 - r_2)^2}{(r_1 + r_2)^2 - r_3^2} = \frac{4r_1r_2}{(r_1 + r_2 + r_3)(r_1 + r_2 - r_3)}.$$

Now combine this with (20) to get

$$s_3 = \frac{4h_3^2}{\left(1 + h_3^2\right)^2} = \frac{(r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)}{4r_1^2r_2^2}.$$

So

$$\frac{(S(h_3))^2}{r_3^2} = \frac{4h_3^2}{\left(1 + h_3^2\right)^2 r_3^2} = \frac{(r_1 + r_2 + r_3)(-r_1 + r_2 + r_3)(r_1 - r_2 + r_3)(r_1 + r_2 - r_3)}{4r_1^2r_2^2r_3^2}.$$

But this is symmetric in the three indices, so that

$$\frac{(S(h_1))^2}{r_1^2} = \frac{(S(h_2))^2}{r_2^2} = \frac{(S(h_3))^2}{r_3^2} = \frac{\mathcal{A}}{4r_1^2r_2^2r_3^2}.$$

Now take square roots to get the result, since if one relative half-turn is positive, the others are also. $\blacksquare$

Theorem 15 (Triple turn formula) *For any three vectors $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$, suppose that*

$$h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2) \qquad h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3) \qquad \text{and} \qquad h_{31} \equiv h(\mathbf{v}_3, \mathbf{v}_1).$$

Then

$$h_{12} + h_{23} + h_{31} = h_{12}h_{23}h_{31}.$$

Proof. If $h_1 \equiv h(\mathbf{v}_1)$, $h_2 \equiv h(\mathbf{v}_2)$ and $h_3 \equiv h(\mathbf{v}_3)$ then the result follows from the identity

$$\frac{h_3 - h_2}{1 + h_2h_3} + \frac{h_1 - h_3}{1 + h_3h_1} + \frac{h_2 - h_1}{1 + h_1h_2} = \left(\frac{h_3 - h_2}{1 + h_2h_3}\right)\left(\frac{h_1 - h_3}{1 + h_3h_1}\right)\left(\frac{h_2 - h_1}{1 + h_1h_2}\right). \quad \blacksquare$$

As a consequence, if two of the half-turns h_{12}, h_{23}, h_{31} are known, we get *a linear equation for the third.* The next result is the rotor analog of the fact that the angles of a triangle add to π, using the notation of Figure 2.

Theorem 16 (Triangle turn formula) *Suppose that* $\overrightarrow{A_1A_2A_3}$ *is an oriented triangle with half turns*

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right) \qquad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \qquad \text{and} \qquad h_3 \equiv h\left(\overrightarrow{A_3A_1}, \overrightarrow{A_3A_2}\right).$$

Then

$$h_1h_2 + h_1h_3 + h_2h_3 = 1.$$

Proof. Apply the previous result to the vectors $\mathbf{v}_1 \equiv \overrightarrow{A_1A_2}$, $\mathbf{v}_2 \equiv \overrightarrow{A_2A_3}$ and $\mathbf{v}_3 \equiv \overrightarrow{A_3A_1}$, so that $h_{12} = -1/h_2$, $h_{23} = -1/h_3$ and $h_{31} = -1/h_1$. Then

$$-\frac{1}{h_2} - \frac{1}{h_3} - \frac{1}{h_1} = -\frac{1}{h_1h_2h_3}.$$

After clearing denominators, this becomes

$$h_1h_2 + h_1h_3 + h_2h_3 = 1. \blacksquare$$

We may now *analyse triangles completely accurately,* without relying either on the usual $30°, 45°, 60°$ or $90°$ formulas, or approximate values obtained for the circular functions by our calculators.

11 Quadrilateral formulas

While the previous two theorems have different formulas, the situation for four points is more symmetric.

Theorem 17 (Quadruple turn formula) *For any four vectors* $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ *and* $\mathbf{v}_4$, *suppose that*

$$h_{12} \equiv h(\mathbf{v}_1, \mathbf{v}_2) \qquad h_{23} \equiv h(\mathbf{v}_2, \mathbf{v}_3) \qquad h_{34} \equiv h(\mathbf{v}_3, \mathbf{v}_4) \qquad \text{and} \qquad h_{41} \equiv h(\mathbf{v}_4, \mathbf{v}_1).$$

Then

$$h_{12} + h_{23} + h_{34} + h_{41} = h_{12}h_{23}h_{34} + h_{12}h_{23}h_{41} + h_{12}h_{31}h_{41} + h_{23}h_{34}h_{41}.$$

Proof. We suppose that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ have half-turns h_1, h_2, h_3 and h_4 respectively. After factoring out $h_{12}h_{23}h_{34}h_{41}$ from the right hand side, the required result is a consequence of the identity

$$\begin{aligned} &\frac{h_2 - h_1}{1 + h_1h_2} + \frac{h_3 - h_2}{1 + h_2h_3} + \frac{h_4 - h_3}{1 + h_3h_4} + \frac{h_1 - h_4}{1 + h_4h_1} \\ = &\left(\frac{h_2 - h_1}{1 + h_1h_2}\right)\left(\frac{h_3 - h_2}{1 + h_2h_3}\right)\left(\frac{h_4 - h_3}{1 + h_3h_4}\right)\left(\frac{h_1 - h_4}{1 + h_4h_1}\right)\left(\frac{1 + h_1h_2}{h_2 - h_1} + \frac{1 + h_2h_3}{h_3 - h_2} + \frac{1 + h_3h_4}{h_4 - h_3} + \frac{1 + h_4h_1}{h_1 - h_4}\right). \blacksquare \end{aligned}$$

Theorem 18 (Quadrilateral turn formula) *Suppose that* $\overrightarrow{A_1A_2A_3A_4}$ *is an oriented quadrilateral with half turns*

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_4}\right) \qquad h_2 \equiv h\left(\overrightarrow{A_2A_3}, \overrightarrow{A_2A_1}\right) \qquad h_3 \equiv h\left(\overrightarrow{A_3A_4}, \overrightarrow{A_3A_2}\right) \qquad \text{and} \qquad h_4 \equiv h\left(\overrightarrow{A_4A_1}, \overrightarrow{A_4A_3}\right).$$

Then

$$h_1 + h_2 + h_3 + h_4 = h_1h_2h_3 + h_1h_2h_4 + h_1h_3h_4 + h_2h_3h_4.$$

Proof. Apply the previous result to the vectors $v_1 \equiv \overrightarrow{A_1A_2}$, $v_2 \equiv \overrightarrow{A_2A_3}$, $v_3 \equiv \overrightarrow{A_3A_4}$ and $v_4 \equiv \overrightarrow{A_4A_1}$, so that $h_{12} = -1/h_2$, $h_{23} = -1/h_3$, $h_{34} = -1/h_4$ and $h_{41} = -1/h_1$. Then

$$-\frac{1}{h_2} - \frac{1}{h_3} - \frac{1}{h_4} - \frac{1}{h_1} = \frac{1}{h_1h_2h_3h_4}(-h_2 - h_3 - h_4 - h_1)$$

and after clearing denominators we get the result. $\blacksquare$

Four points in the plane actually determine six lengths, or better yet six *quadrances* (squares of lengths), since we can also consider the diagonals of a quadrilateral. These six quadrances are not independent. The following result is a special case of a formula of Euler for the volume of a tetrahedron in terms of the quadrances of its sides (see [1]). It is usual to prove it using linear algebra, but here we use vector trigonometry.

Theorem 19 (Four point relation) *Suppose that the triangle $\overline{A_1A_2A_3}$ has quadrances Q_1, Q_2 and Q_3, and that A_4 is any point with quadrances $P_1 \equiv Q(A_1, A_4)$, $P_2 \equiv Q(A_2, A_4)$ and $P_3 \equiv Q(A_3, A_4)$. Then*

$$E(Q_1, Q_2, Q_3, P_1, P_2, P_3) \equiv \det \begin{pmatrix} 2P_1 & P_1 + P_2 - Q_3 & P_1 + P_3 - Q_2 \\ P_1 + P_2 - Q_3 & 2P_2 & P_2 + P_3 - Q_1 \\ P_1 + P_3 - Q_2 & P_2 + P_3 - Q_1 & 2P_3 \end{pmatrix} = 0.$$

Proof. We frame the argument in terms of lengths and convert to quadrances at the end. Suppose that the triangle $\overline{A_1A_2A_3}$ has lengths d_1, d_2 and d_3 as in Figure 7, and that the distances from A_4 to the points A_1, A_2 and A_3 are respectively r_1, r_2 and r_3. Define the half-turns

$$h_1 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_3}\right) \qquad h_2 \equiv h\left(\overrightarrow{A_1A_3}, \overrightarrow{A_1A_4}\right) \qquad \text{and} \qquad h_3 \equiv h\left(\overrightarrow{A_1A_2}, \overrightarrow{A_1A_4}\right).$$

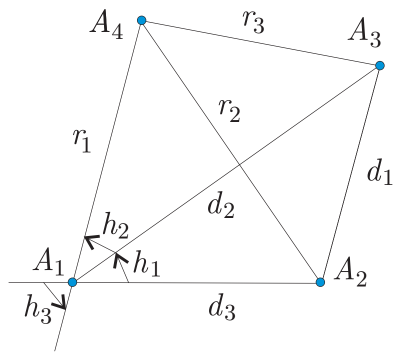

Figure 7: Four planar points

From the Cross law

$$C_1 \equiv C(h_1) = \frac{d_2^2 + d_3^2 - d_1^2}{2d_2d_3} \qquad C_2 = C(h_2) = \frac{d_2^2 + r_1^2 - r_3^2}{2d_2r_1} \qquad C_3 = C(h_3) = \frac{d_3^2 + r_1^2 - r_2^2}{2d_3r_1}.$$

Now apply the Triple C formula:

$$C_1^2 + C_2^2 + C_3^2 = 1 + 2C_1C_2C_3$$

to get

$$\begin{aligned} & \left(\frac{d_2^2 + d_3^2 - d_1^2}{2d_2d_3}\right)^2 + \left(\frac{d_2^2 + r_1^2 - r_3^2}{2d_2r_1}\right)^2 + \left(\frac{d_3^2 + r_1^2 - r_2^2}{2d_3r_1}\right)^2 \\ = \quad & 1 + 2\left(\frac{d_2^2 + d_3^2 - d_1^2}{2d_2d_3}\right)\left(\frac{d_2^2 + r_1^2 - r_3^2}{2d_2r_1}\right)\left(\frac{d_3^2 + r_1^2 - r_2^2}{2d_3r_1}\right). \end{aligned}$$

Clearing the denominators and simplifying gives

$$\begin{aligned} 0 \quad = \quad & d_1^2r_1^4 + d_1^4r_1^2 + d_2^2r_2^4 + d_2^4r_2^2 + d_3^2r_3^4 + d_3^4r_3^2 + d_1^2d_2^2d_3^2 - d_1^2d_2^2r_1^2 - d_1^2d_2^2r_2^2 - d_1^2d_3^2r_1^2 - d_1^2d_3^2r_3^2 - d_2^2d_3^2r_2^2 \\ & - d_2^2d_3^2r_3^2 - d_1^2r_1^2r_2^2 - d_1^2r_1^2r_3^2 - d_2^2r_1^2r_2^2 + d_1^2r_2^2r_3^2 + d_2^2r_1^2r_3^2 + d_3^2r_1^2r_2^2 - d_2^2r_2^2r_3^2 - d_3^2r_1^2r_3^2 - d_3^2r_2^2r_3^2. \end{aligned}$$

All the terms are square, and so using $Q_1 \equiv d_1^2$, $Q_2 \equiv d_2^2$, $Q_3 \equiv d_3^2$ and $P_1 \equiv r_1^2$, $P_2 \equiv r_2^2$, $P_3 \equiv r_3^2$ we get, up to a factor of $-1/2$, the relation

$$\det \begin{pmatrix} 2P_1 & P_1 + P_2 - Q_3 & P_1 + P_3 - Q_2 \\ P_1 + P_2 - Q_3 & 2P_2 & P_2 + P_3 - Q_1 \\ P_1 + P_3 - Q_2 & P_2 + P_3 - Q_1 & 2P_3 \end{pmatrix} = 0. \quad \blacksquare$$

12 Area and integration with rotor coordinates

The Jacobian matrix for the transformation (6) is

$$J=\begin{pmatrix}\frac{1-h^2}{1+h^2} & \frac{2h}{1+h^2}\\ \frac{-4rh}{(1+h^2)^2} & \frac{2r\left(1-h^2\right)}{(1+h^2)^2}\end{pmatrix}$$

with

$$\det J=\frac{2r}{1+h^2}=rM\left(h\right). \tag{22}$$

The area of a circle of radius R is then

$$\int_0^R\int_0^\infty\frac{2r}{1+h^2}dhdr=\pi R^2$$

where

$$\pi\equiv\int_{-\infty}^{\infty}\frac{1}{1+h^2}dh=\frac{1}{2}\int_{-\infty}^{\infty}M\left(h\right)dh$$

may be introduced as the definite integral of a rational function. This approach to π has the advantage of requiring no prior theory of Euclidean geometry or knowledge of transcendental circular functions.

The rotational invariant measure on the circle can be computed from (22) or from the formula (12) for the circle sum; it is

$$d\mu=\frac{2}{1+h^2}dh=M\left(h\right)dh. \tag{23}$$

The measure $d\mu$ is normalized so that for small h, $d\mu\approx 2dh$ in accordance with the length of an approximating (vertical) linear segment. The total measure of the circle is 2π.

For a rational function $f\left(x,y\right)$ the integral $I=\int_{c_U}f\left(x,y\right)\ d\sigma$ over the unit circle can then be evaluated by expressing it as

$$I=\int_{(-\infty,\infty)}f\left(C\left(h\right),S\left(h\right)\right)M\left(h\right)dh. \tag{24}$$

Since this is a definite integral of a rational function of h, it may be evaluated using well-known techniques. In contract, with polar coordinates the integral

$$I=\int_{[0,2\pi]}f\left(\cos\theta,\sin\theta\right)d\theta$$

is not easy to solve directly with elementary means, and is usually transformed via the $\tan\theta/2$ substitution into exactly (24). This is further support that the rotor form has an intrinsic fundamental aspect.

13 Kinematics in rotor coordinates

We now recast some basic aspects of kinematics in terms of vector trigonometry. If

$$\mathbf{e}_1=\left(C\left(h\right),S\left(h\right)\right)\qquad\text{and}\qquad\mathbf{e}_2=\left(-S\left(h\right),C\left(h\right)\right) \tag{25}$$

are perpendicular unit vectors, with h a function of time t, then from the C and S derivative theorem

$$\frac{d\mathbf{e}_1}{dt}=M\left(h\right)\dot{h}\,\mathbf{e}_2\qquad\text{and}\qquad\frac{d\mathbf{e}_2}{dt}=-M\left(h\right)\dot{h}\,\mathbf{e}_1 \tag{26}$$

where we use the physicists' notation

$$\dot{h}\equiv\frac{dh}{dt}.$$

Suppose that a particle P moves in a plane so that at each point its position vector $\mathbf{p}\equiv\mathbf{p}\left(t\right)$ is a multiple $r=r\left(t\right)$ of a unit vector $\mathbf{e}_1$, with $\mathbf{e}_2$ the perpendicular unit vector as in (25). Thus h is also a function of t. This is shown in Figure 8.

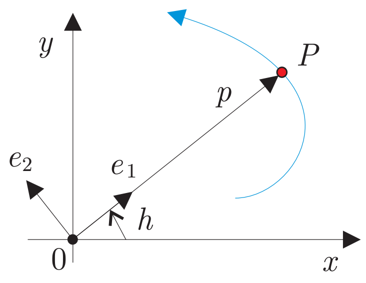

Figure 8: Position of a moving particle

Differentiate

$$\mathbf{p} = r\,\mathbf{e}_1$$

and use (26), to find the velocity

$$\begin{aligned}\mathbf{v} &= \frac{d\mathbf{p}}{dt} = \dot{r}\,\mathbf{e}_1 + r\frac{d\mathbf{e}_1}{dt} \\ &= \dot{r}\,\mathbf{e}_1 + rM(h)\,\dot{h}\,\mathbf{e}_2. \qquad (27)\end{aligned}$$

The acceleration is, after some rewriting using again (26),

$$\begin{aligned}\mathbf{a} &= \frac{d\mathbf{v}}{dt} = \ddot{r}\,\mathbf{e}_1 + \dot{r}\frac{d\mathbf{e}_1}{dt} + \frac{d}{dt}\left(rM(h)\,\dot{h}\right)\mathbf{e}_2 + rM(h)\,\dot{h}\frac{d\mathbf{e}_2}{dt} \\ &= \left(\ddot{r} - rM^2(h)\,\dot{h}^2\right)\mathbf{e}_1 + \frac{1}{r}\frac{d}{dt}\left(r^2M(h)\,\dot{h}\right)\mathbf{e}_2. \qquad (28)\end{aligned}$$

The two equations (27) and (28) give the resolution of the velocity and acceleration along and perpendicular to the position vector.

14 Central forces

Suppose now that the moving particle P with position vector $\mathbf{p} = \mathbf{p}(t)$ is subject to a central force

$$\mathbf{F}(\mathbf{p}) \equiv -F(r(\mathbf{p}))$$

centered at the origin O. Newton's equation of motion is $\mathbf{F} = m\,\mathbf{a}$. If $\mathbf{p}_0$ and $\mathbf{v}_0$ are initial values at time $t = 0$ of the position and velocity, then the entire path of the particle will lie in the plane spanned by these vectors.

Now let r and h be rotor coordinates of P in this plane, both functions of t, so that

$$\mathbf{p} = r\,\mathbf{e}_1$$

where $\mathbf{e}_1 \equiv (C(h), S(h))$ is a (moving) unit vector, and $\mathbf{e}_2 \equiv (-S(h), C(h))$ is a second unit vector perpendicular to $\mathbf{e}_1$. Equation (28) shows that

$$-F = m\left(\ddot{r} - rM^2(h)\,\dot{h}^2\right) \qquad (29)$$

$$0 = \frac{d}{dt}\left(r^2M(h)\,\dot{h}\right). \qquad (30)$$

It follows that for a central force,

$$r^2M(h)\,\dot{h} = c \qquad (31)$$

for some constant c. This is the *Law of Conservation of Angular Momentum* in rotor coordinates.

We wish to solve the equations (29) and (30) to relate r and h. It is a standard trick (see for example Richmond [4]) to define a new variable

$$w \equiv \frac{1}{r}$$

so that

$$\dot{h} = \frac{cw^2}{M(h)}. \tag{32}$$

Then

$$\dot{r} = -\frac{\dot{w}}{w^2} = -\frac{1}{w^2}\frac{dw}{dh}\dot{h} = -\frac{c}{M(h)}\frac{dw}{dh} \tag{33}$$

and

$$\ddot{r} = \frac{c\dot{h}}{M^2(h)}\frac{dM}{dh}\frac{dw}{dh} - \frac{c}{M(h)}\dot{h}\frac{d^2w}{dh^2} = -c\dot{h}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{dw}{dh}\right).$$

Substitute into (29) and replace $\dot{h}$ using (32) to obtain

$$-\frac{F}{m} = -\frac{c^2w^2}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{dw}{dh}\right) - \frac{1}{w}M^2(h)\left(\frac{cw^2}{M(h)}\right)^2$$

or

$$\frac{1}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{dw}{dh}\right) + w = \frac{F}{c^2mw^2}. \tag{34}$$

Since F is a function of r, and so of w, this is a differential equation describing $w = r^{-1}$ in terms of the half-turn h.

15 Inverse square force and Kepler-Newton orbits

Now suppose that $\mathbf{F}$ is an *inverse square force*. Then the right hand side of (34) is a constant $k > 0$, and

$$\frac{1}{M(h)}\frac{d}{dh}\left(\frac{1}{M(h)}\frac{dw}{dh}\right) + w = k.$$

By the C and S second order derivative theorem, solutions to the homogeneous case ($k = 0$) are given by $w(h) = C(h)$ and $w(h) = S(h)$, while a particular solution is obviously the constant $w = k$. So the general solution, now in terms of the original rotor variables r and h, is the linear combination

$$\frac{1}{r} = aC(h) + bS(h) + k \tag{35}$$

where a and b are constants that depend on initial conditions. We will now show that this is a conic by transforming the equation to x and y coordinates.

Use $C(h) = x/r$ and $S(h) = y/r$ to get

$$\frac{1 - ax - by}{r} = k$$

so that

$$(1 - ax - by)^2 = k^2\left(x^2 + y^2\right) \tag{36}$$

which we recognize as a second order equation in x and y, hence a conic section. If we divide by $a^2 + b^2$, the left hand side is the quadrance (square of distance) from the point (x, y) to the line d with equation $ax + by = 1$. So (36) states that the ratio of the quadrance from $P \equiv (x, y)$ to the focus $O \equiv (0, 0)$ to the quadrance from P to the directrix d is

$$e^2 = \frac{a^2 + b^2}{k^2}.$$

If $e^2 > 1$ or equivalently $a^2 + b^2 > k^2$ then the conic is a hyperbola, if $e^2 = 1$ or equivalently $a^2 + b^2 = k^2$ then the conic is a parabola, and if $e^2 < 1$ or equivalently $a^2 + b^2 < k^2$ then the conic is an ellipse. This approach does not require transcendental functions or knowledge of pedal equations of conics.

16 The parabolic case

In this section we look at the particular attractive and amenable special case of parabolic motion, which connects naturally with the geometry of this remarkable conic. Most comets have orbits which are nearly parabolic.

Choose $a = k = 1$ and $b = 0$ so that (35) becomes

$$\frac{1}{r} = \frac{1-h^2}{1+h^2} + 1 = M(h) \qquad \text{or} \qquad r = \frac{1+h^2}{2}$$

while (36) gives the Cartesian equation

$$(1-x)^2 = x^2 + y^2 \qquad \text{or} \qquad y^2 = 1 - 2x.$$

This is a parabola with focus O and directrix the line $x = 1$.

The equation for Conservation of angular momentum gives, after normalizing so that $c = 1$,

$$\frac{dh}{dt} = M(h) = \frac{2}{1+h^2}.$$

This is an easy integration, and we find that

$$\int 1 + h^2 \, dh = h + \frac{h^3}{3} = \int 2 \, dt = 2t$$

where we initialize our time t so that $t = 0$ corresponds to $h = 0$. Then also we may deduce that for a point $P \equiv (x, y)$ on the orbit,

$$\begin{aligned} x &= r\left(\frac{1-h^2}{1+h^2}\right) = \frac{1}{M(h)}\left(\frac{1-h^2}{1+h^2}\right) = \frac{1-h^2}{2} = 1 - r \\ y &= r\left(\frac{2h}{1+h^2}\right) = \frac{1}{M(h)}\left(\frac{2h}{1+h^2}\right) = h. \end{aligned}$$

So the y coordinate of P equals its half-turn h, giving the half-turn direct physical significance. This relates to a standard construction of the parabola: from a point E on the unit circle join $(-1, 0)$ to the y-axis at the point $(0, h)$, then P is the meet of the lines $y = h$ and OE.

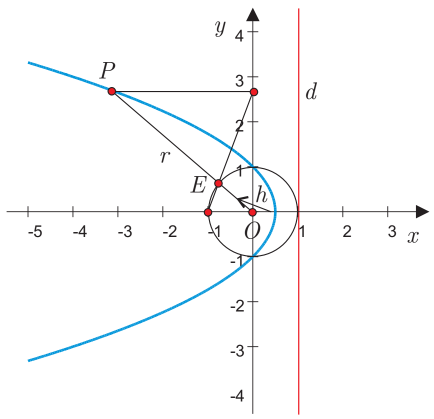

Figure 9: A parabolic trajectory

The velocity is also of interest, and

$$\frac{d}{dh}(x, y) = (-h, 1)$$

so that

$$\mathbf{v} = \frac{d}{dt}(x, y) = \left(\frac{-2h}{1+h^2}, \frac{2}{1+h^2}\right) = (\mathbf{v}_x, \mathbf{v}_y).$$

Note that $\mathbf{v}_x = -h\mathbf{v}_y$, and that $\mathbf{v}$ lies on the **velocity circle** c_v with equation $x^2 + (y-1)^2 = 1$. It is a general fact that for any orbit the velocities lie on a suitable circle, but in this case there is a direct connection between the velocity at P and P itself, since $\mathbf{v}$ is perpendicular to the line OF through the foot of the perpendicular from P to the directrix d, as in Figure 10. The reader is also reminded that the tangent to the orbit at P meets this segment $\overline{OF}$ perpendicularly at its midpoint M, giving another construction of the parabola.

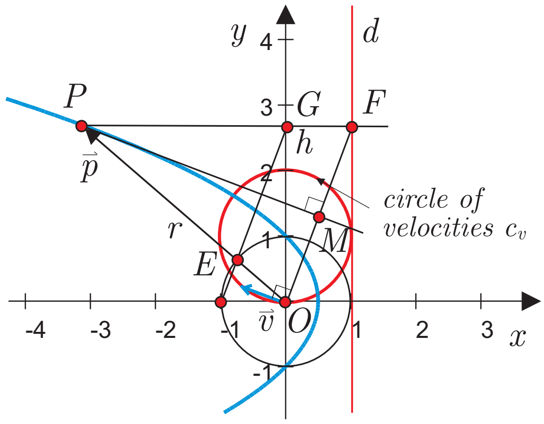

Figure 10: Parabolic motion and the velocity circle c_v (red)

A computation then shows that, using rotor coordinates,

$$\mathbf{v} = (\mathbf{v}_x, \mathbf{v}_y) = \left| \frac{2}{\sqrt{1+h^2}}, h + \sqrt{1+h^2} \right\rangle.$$

For elliptic or hyperbolic orbits, finding h and r as functions of t is more work, but the idea is the same.

In summary, rotor coordinates and vector trigonometry give a powerful new—yet familiar—technology for both practical and theoretical problems involving planar geometry; and planetary motion is a topic that can be understood and appreciated by first year undergraduates. It is time for another look at mathematics education, this time with a focus on *trigonometry* and *geometry*, and their wider and richer applications.

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