Let A be a monoid. The submonoids $B, C \leq A$ are **mutually singular** [1], in symbols $B \perp C$, if

- $B \cap C = \{1\}$, $A = \langle B \cup C \rangle$,
- for all $b \in B$ and $a, a' \in A$, $aba' \in C \implies b = 1$, and
- for all $c \in C$ and $a, a' \in A$, $aca' \in B \implies c = 1$.

Let A, B, and C be as above. If the binary relation

$$\{(\prod_{i=1}^{n} b_i c_i, \prod_{i=1}^{n} b_i) \colon \bar{b} \in B, \bar{c} \in C\} = \pi_B^C \subseteq A \times A$$

is a function $\pi_B^C \colon A \to B$, then it is the B**-filter along** C. It is left as an exercise for the reader to prove that it is a an idempotent homomorphism when it exists. If $\pi_B^C(a) = \pi_B^{C'}(a)$ for all C, C' such that $B \perp C, C'$ and all $a \in A$, then it is π_B, the B**-filter**.

References

[1] Folland "Real Analysis" (1984) p. 82.