- define sector, calc

- define differential action, disc, base of a disc

- define automorphism group of a differential action, let $C' \leq C$ be a subsector. Let $Y \subseteq X$ be any subset. The galois group of the differential action is $\mathrm{Aut}_{C'}(X/Y) = \{\sigma\colon \sigma\colon X \leftrightarrow X,\ \sigma(xc') = \sigma(x)c',\ d(\sigma(x), \sigma(xc')) = d(x, xc'),\ \sigma(y) = y\ \forall c' \in C', y \in Y\}$. This automorphism group gives us two "dials" to control the level of detail: the size of C' and the size of Y. The disc joins these by suggesting the choice of $Y = bC'$ for some fixed base point $b \in X$.

Let C, D be sectors and X a right C-set. Let $d\colon X \rtimes C \to D$ be a differential action. A *disc* is a pair (d, b) such that $b \in X$ and $X = bC$, and the ordered pair (X, b) is denoted X^b. Let $C' \leq C$ be a subsector, and $Y \subseteq X$ a subset such that $b \in Y$. The galois group of the disc is $\mathrm{Aut}_{C'}(X^b/Y) = \{\sigma \in \mathrm{Aut}_{C'}(X/Y)\colon \sigma(b) = b\}$. Note that since $\sigma(bc') = \sigma(b)c' = bc'$, we have $\mathrm{Aut}_{C'}(X^b/Y) = \mathrm{Aut}_{C'}(X^b/(Y \cup bC'))$. When Y is not given, let $\mathrm{Aut}_{C'}(X^b) = \mathrm{Aut}_{C'}(X^b/bC')$, and when C' is not given, let $\mathrm{Aut}(X^b/Y) = \mathrm{Aut}_1(X^b/Y)$.

- define right differential action, right regular disc for a calc

- define right shift action for a calc

Example. Let k be a positive natural number, and let X be the right differential action, X^b the right regular disc of the calc $e\colon \mathbb{N} \to \mathbb{N}$ defined by $e(n) = \lfloor n/k \rfloor$. Note that the right shift action cannot be turned into a disc (d, b) such that $e(n) = d(b, bn)$ because the cardinality of the range of e is countably infinite whereas the the cardinality of the range of d is no greater than k^2. It is left as an exercise for the reader to show that $\mathrm{Aut}(X) \cong S_k^{\mathbb{N}}$ and $\mathrm{Aut}(X^b) \cong S_{k-1} \times S_k^{\mathbb{N}}$.

From this point of view, the right regular disc for e represents the cycles of a rotating cam, with units emitted counting complete revolutions. A disc automorphism creates a "zig-zag" pattern through each of the segments $[0, k-1]$, $[k, 2k-1], \ldots$ named by an element of S_k for all but the first, an element of S_{k-1}. For the right regular differential action, they're all S_k.

Example. Finite step indicators. Let e be as in the last example and let $f(n) = \max(1, e(n))$. In this case, the right shift action can be turned into a disc, letting $d(f^i, f^j) = 1_{j=k \wedge i<k}$. Let this disc be Z with base f, and let X^b be the right regular disc for e. It is left as an exercise for the reader to show that $\mathrm{Aut}(X^b) \cong \mathrm{Aut}(Z/f) \cong S_{k-1}$.

Cascade concatenation operation for two calcs, a finite tuple of calcs, and an infinite sequence of calcs when the domains are all sectors and a relay signal symbol is chosen to switch input over to the next calc in sequence

Experience from the last examples shows that we can count to k and light an indicator lamp, and that the symmetry group for the differential action is S_k. If, instead, one lamp is lit after k_1 steps and another after k_2, then we can mix up the k_1 lower states and the k_2 higher states as long as transitioning from a lower to a higher lights a lamp. This yields an automorphism group of $S_{k_1} \times S_{k_2}$ for the differential action. In order to construct this, employ a general mechanic yielding the cascade concatenation of two calcs.

Let C be a sector. A nonempty subset $P \subseteq C$ has the *prefix* property if for all $p, q \in P$, $p \in qC \implies p = q$. Write $c|P$ for $cC \cap P \neq \emptyset$ and $P|c$ for $c \in PC$, and note that $c|P \wedge P|c \implies c \in P$ when P is prefix.

Let C be a free monoid, and let $|c|$ be the *length* of c, the sum of the generators in the written representation of c, weighted by number of occurrences.

Prop. Let C be a free monoid and $c \in C$. If $c \in p_1C \cap p_2C$ for $p_i \in C$, then either $p_1 \in p_2C$ or $p_2 \in p_1C$.

Proof. Left as an exercise for the reader. (Hint: split into cases by $|p_1| \leq |p_2|$ or not.)

Let C be a sector. The nonempty prefix subset $T \subseteq C$ has the *threshold* property if $c|T \vee T|c$ for all $c \in C$.

Let C, D be free monoids, and let $T \subseteq C$ be a nonempty threshold subset. If $T|c$ and $c \in t_1C \cap t_2C$, then either $t_1 \in t_2C$ or $t_2 \in t_1C$, so $t_1 = t_2$. Define $\cdot_{|T} \colon C \to C$, the *restriction map* to T, as

$$c_{|T} = \begin{cases} c & \text{if } c|T \\ \text{the unique } t \in T \text{ such that } c \in tC & \text{if } T|c \end{cases}.$$

Define $T^- \colon C \to C$, left *removal*, as

$$T^-c = \begin{cases} t^{-1}c & \text{if } T|c, \text{ where } c \in tC \\ c & \text{if } c|T \end{cases}.$$

- define restriction map $\cdot_{|P} \colon C \to C$ and left semidivision $P^- \colon C \to C$

Prop. Let C, D be free monoids and $e, f \colon C \to D$ calcs. Let $T \subseteq C$ be a nonempty threshold subset. Then the map $g \colon C \to D$ defined by $g(c) = e(c_{|T})f(T^-c)$ is a calc.

Intuitively, the teacup T runneth over, spilling out what would have gone to e into f's teacup.