A Closed-Form Solution to the Geometric Goat Problem

INGO ULLISCH

The geometric goat problem (not to be confused with the goat problem in probability theory) is a generic name for two different geometric problems in recreational mathematics: the interior goat problem and the exterior goat problem. Although only the former is dealt with in this paper, let us briefly review their respective historical origins for the sake of completeness. According to the article on the goat problem in Wikipedia,[1] the exterior goat problem dates back to 1748, when it was first published in *The Ladies' Diary: or, Woman's Almanack* [23, p. 41]. This annual mathematical periodical featured, among other things, calendrical information, including times of sunrise and sunset and the phases of the moon, along with a list of mathematical questions the solutions to which were published in the subsequent issue. Reading the original text of the 1748 issue shows that in the course of history, a horse was transformed into a goat (see Figure 1):

> VIII. QUESTION 302. By Upnorensis.
>
> Observing a Horse tied to feed in a Gentleman's Park, with one End of a Rope to his Fore-foot, and the other End to one of the Circular Iron-Rails, inclosing a Pond, the Circumference of which Rails being 160 Yards, equal to the Length of the Rope, what Quantity of Ground, at most, could the Horse feed?

Besides the solution in the 1749 issue of the periodical [6, p. 25], a solution for the more general case in which the circular iron rails are replaced by a smooth convex curve was given in [9].[2] According to [3], the interior problem was originally published in the *American Mathematical Monthly*'s first issue [7] in 1894, where the problem is posed as follows (see Figure 2):

> 30. Proposed by Charles E. Myers, Canton, Ohio.
>
> A circle containing one acre is cut by another whose centre is on the circumference of the given circle, and the area common to both is one-half acre. Find that radius of the cutting circle.

More information on both problems can be found in [3], which also provides further references to variations of grazing goat problems; the original solution can be found in [17]. Nowadays, the interior case is the more prominent one. A possible reason is that some interesting generalizations of this problem have been investigated. For instance, instead of circles one can consider n-balls and ask for the radius of the cutting n-ball in an arbitrary dimension n. In [4, 14] it is shown that the radius of the cutting n-ball converges to $\sqrt{2}$ as n tends to infinity. Another reason

[1] Available online at https://en.wikipedia.org/wiki/Goat_problem.

[2] The author uses the term *tethered-bull problem* and does not refer to [23], but to a post in the Internet newsgroup sci.math.

https://doi.org/10.1007/s00283-020-09966-0

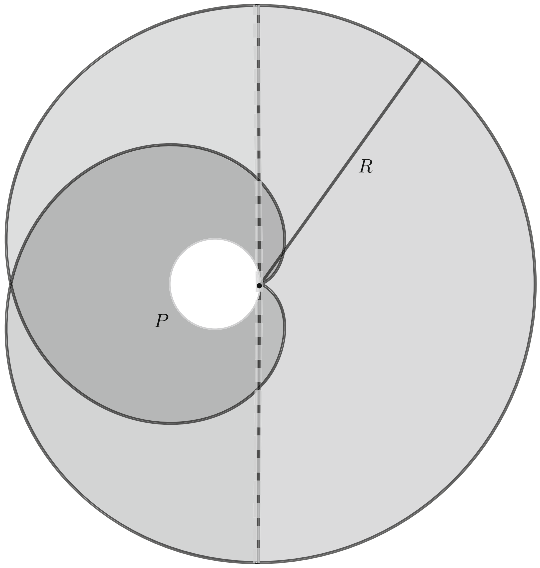

Figure 1. The exterior goat problem. A goat is tethered to a fence surrounding a circular pond P (the white circle), where the length R of the rope equals the circumference of P. The gray shaded region (light and dark gray areas combined) is that in which the goat can graze. It consists of a semicircle and the area bordered by the pond and the interior of two curves, both of which are involutes of P.

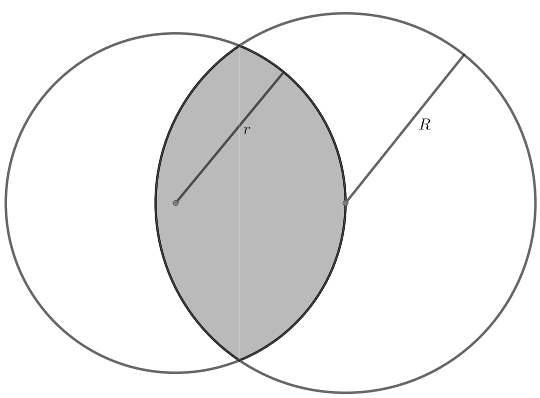

Figure 2. The interior goat problem. A goat is tethered to the boundary of a circular field of radius r. How long must the length R of the rope be chosen such that the goat can graze exactly half the field?

might stem from the fact that it is surprisingly difficult to find a closed-form solution, since the radius of the cutting circle is given as the solution of a transcendental equation. To the best of my knowledge, only approximate solutions have been given previously, using either Newton's method or another method of choice; see, e.g., [4, 7]. After more than 120 years it is time to add a new twist to the story by constructing a closed-form solution to the interior problem. First, we will review the standard approaches to this problem and then explain how to find a closed-form solution by means of elementary complex analysis.

A Closed-Form Solution

Before deriving a closed-form solution, we show that the goat problem is an instance of a common phenomenon in mathematics whereby a straightforward approach to a seemingly easy problem turns out to be unreasonably difficult. In the case of the problem at hand, such a straightforward, if not the most straightforward, approach is to use integration as indicated in Figure 3.

Let $B_j, j = 1, 2$, denote the region bounded by the circle k_j, the x-axis, and the line connecting x_0 and the intersection point of the two circles; see Figure 3. An easy calculation shows that $x_0 = r - R^2/(2r)$, and the area of B_j can be computed via integration. We skip the details here, since they are not relevant to the subsequent construction of a closed-form solution.[3] Doing all the tedious computations yields

$$\begin{aligned}\mathcal{A}(B_2) + \mathcal{A}(B_1) &= \int_{r-R}^{r-\frac{R^2}{2r}} \sqrt{R^2 - (x-r)^2}\, dx \\ &\quad + \int_{r-\frac{R^2}{2r}}^{r} \sqrt{r^2 - x^2}\, dx \\ &= \frac{\pi r^2}{4},\end{aligned}$$

where $\mathcal{A}$ denotes area, which finally results in the transcendental equation

$$\begin{aligned}&\frac{\pi R^2}{4} - \frac{R}{4}\sqrt{4r^2 - R^2} - \frac{R^2}{2}\arcsin\left(\frac{R}{2r}\right) \\ &- \frac{r^2}{2}\arcsin\left(\frac{2r^2 - R^2}{2r^2}\right) \\ &= 0.\end{aligned}$$

Another approach, which is somewhat similar to the previous one, yields a slightly easier term (slightly easier in that it contains only one arcsine term). We merely state the resulting equation and refer the reader to Wikipedia. Here it is:

$$\frac{\pi}{4}(R^2 - r^2) - \frac{R}{4}\sqrt{4r^2 - R^2} + \left(r^2 - \frac{R^2}{2}\right)\arcsin\left(\frac{R}{2r}\right) = 0.$$

These equations do not appear to have a closed-form solution. It turns out that a first step toward such a solution is to find an expression for the area of the intersection of the circles that does not depend on the radii. Let us take a look at Figure 4, and let A_1 denote the portion of the disk enclosed by the circle k_1 lying to the right of the chord QQ'. Similarly, A_2 denotes the portion of the disk enclosed by k_2 lying to the left of the chord QQ'. Applying the formula for calculating the area of a circular segment, we have

$$\mathcal{A}(A_1) = \frac{r^2}{2}(\alpha - \sin\alpha) \quad \text{and} \quad \mathcal{A}(A_2) = \frac{R^2}{2}(\beta - \sin\beta).$$

We have the following lemma.

[3] The interested reader can find the detailed calculations at http://www.bigbandi.de/dokus/ziege/index.html.

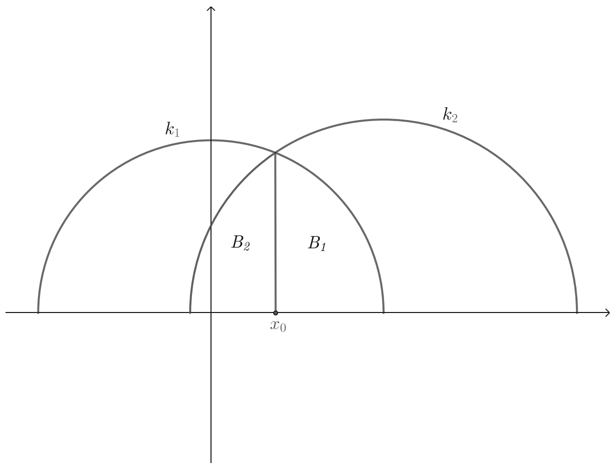

Figure 3. Using integration to solve the goat problem, where the circle k_1 represents the boundary of the field.

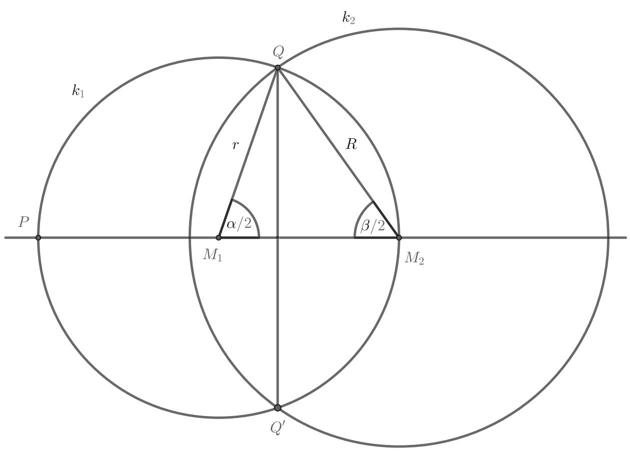

Figure 4. Geometric diagram for the closed-form solution.

Lemma 1. [3] *With the above notation, we have*

1. $\alpha = 2\pi - 2\beta$ *and* $R = 2r\cos(\beta/2)$,
2. $\mathcal{A}(A_2) = r^2(\beta + \beta\cos\beta - \sin\beta - \sin\beta\cos\beta)$,
   $\mathcal{A}(A_1) = r^2(\pi - \beta + \sin\beta\cos\beta)$.

Proof. Consider the triangle M_1M_2Q in Figure 4. Since it is isosceles, we have $\measuredangle M_1QM_2 = \beta/2$, whence

$$\frac{\alpha}{2} + \frac{\beta}{2} + \frac{\beta}{2} = \pi.$$

Consequently, $\alpha = 2\pi - 2\beta$ and

$$\begin{aligned}\mathcal{A}(A_1) &= \frac{r^2}{2}(2\pi - 2\beta - \sin(2\pi - 2\beta))\\ &= r^2(\pi - \beta + \sin\beta\cos\beta).\end{aligned}$$

By Thales's theorem, triangle PM_2Q is a right triangle, so $R = 2r\cos(\beta/2)$. Because of $\mathcal{A}(A_2) = R^2(\beta - \sin\beta)/2$, we then get

$$\begin{aligned}\mathcal{A}(A_2) &= \frac{(2r\cos(\beta/2))^2}{2}(\beta - \sin\beta)\\ &= r^2(1+\cos\beta)(\beta - \sin\beta)\\ &= r^2(\beta + \beta\cos\beta - \sin\beta - \sin\beta\cos\beta),\end{aligned}$$

where we have used the well-known identity

$$\cos\left(\frac{\varphi}{2}\right) = \sqrt{\frac{1+\cos\varphi}{2}}.$$

$\square$

Thus we have reduced the goat problem to finding the solution of the equation

$$\sin\beta - \beta\cos\beta - \frac{\pi}{2} = 0.$$

This equation seems much more tractable than the two above, although it is still transcendental. In the past, there have been attempts to find closed-form solutions of transcendental equations, since the problem of finding the zeros of transcendental functions is encountered in engineering applications such as heat transfer [5, 15, 19, 20] and even quantum mechanics [21]. One of the first methods given is due to Burniston and Siewert [1]. It is based on the theory of singular integral equations as developed by Muskhelishvili in [16]. The crucial part of their method lies in "establishing the appropriate Riemann problem and making use of several elementary properties of the resulting solution to deduce roots of the given transcendental equation."

With regard to the goat problem, their approach seems to be promising, since the exact solution of the equation $\tan\beta = \omega\beta$ is given in [1], where ω is real, which is very close to our problem. However, a disadvantage of this method is that it is unstable under "adding constants." By this we mean that a solution of an equation of the form $f(\beta) = 0$ might be derived by means of this method, whereas in the corresponding "inhomogeneous" case $f(\beta) = c$, $c \neq 0$, it is difficult to implement and leads to expressions that cannot be evaluated in a straightforward manner. It turns out that this is the case with $f(\beta) = \sin\beta - \beta\cos\beta$ and $c = \pi/2$. Instead, we will apply an approach that is surprisingly easy to implement, provided that the transcendental function is analytic and the multiplicity of its zero is low. According to [12], this method was first proposed by Jackson [10, 11] and rediscovered in [13]. It can be stated as the following theorem.

Theorem 1. *Let $U \subseteq \mathbb{C}$ be an open simply connected subset and $f: U \longrightarrow \mathbb{C}$ a nonzero analytic function. For every simple zero $z_0 \in U$ of f, there is a closed curve C in U such that*

$$z_0 = \frac{\oint_C \frac{z\,dz}{f(z)}}{\oint_C \frac{dz}{f(z)}}.$$

PROOF. If $z_0 \in U$ is a zero of f, there exist $\varepsilon > 0$ and an open simply connected subset $U' \subseteq U$ such that $\bar{B}(\varepsilon; z_0) \subseteq U'$ and z_0 is the only zero of f in U'.[4] Let C be the closed curve $\partial B(\varepsilon; z_0)$. Obviously, the function $(z - z_0)/f(z)$ is analytic on U', and Cauchy's integral theorem yields

$$\oint_C \frac{z - z_0}{f(z)}\, dz = 0\,.$$

Rearranging the terms completes the proof. □

REMARK 1. Note that in the situation of the above theorem, one can also consider zeros z_0 of higher multiplicity $n \in \mathbb{N}$. In this case, Cauchy's theorem yields a polynomial expression $z_0^n + a_{n-1} z_0^{n-1} + \cdots + a_0 = 0$. From this and the famous Abel–Ruffini theorem it follows that there is always an explicit expression for zeros with multiplicity $n < 5$.

Under the above conditions, the theorem ensures the existence of a curve that can be used to find a closed-form expression for the zero of the given function. As can be seen from the proof, the equation

$$z_0 = \frac{\displaystyle\oint_C \frac{z\,dz}{f(z)}}{\displaystyle\oint_C \frac{dz}{f(z)}}$$

holds for every Jordan curve C such that $z_0 \in U$ is an element of the interior of C and the only zero of f in the union of C and its interior.

Consider now the entire function $f(z) = \sin z - z\cos z - \pi/2$. On the interval $[0, \pi]$, f increases monotonically from $-\pi/2$ to $\pi/2$, so there is a unique zero z_0 on the interval. Since $f'(z) = z \sin z$, z_0 has multiplicity 1, and Theorem 1 applies. Thus there is a curve that can be used to derive a closed-form expression for z_0. For the sake of completeness and to evaluate this closed-form expression we want to specify a curve that meets the requirement in the second paragraph of Remark 1. In other words, we have to find an enclosure for z_0. Finding zeros of holomorphic functions along with suitable enclosures is a well-studied problem, e.g., [2, 8, 18, 22]. An important basis for this is the argument principle, which states that for a nonconstant meromorphic function g in $R \subseteq \mathbb{C}$ not having any zeros or poles on the simple closed counterclockwise oriented contour ∂R, we have

$$\frac{1}{2\pi \mathrm{i}} \int_{\partial R} \frac{g'(z)}{g(z)}\, dz = N - P\,,$$

where N and P are the respective numbers of zeros and poles of g inside R, counted with multiplicity. If only analytic functions are under consideration, then $P = 0$, and the above integral equals the number of zeros of the function. However, we are in the fortunate situation that there is no need to draw on such techniques. Instead, it is possible to show the desired result using elementary inequalities and some well-known Taylor series expansions.

LEMMA 2. *The function f is nonvanishing on the boundary of the square $R = [\pi/2, \pi] \times \mathrm{i}[-\pi/4, \pi/4]$, and z_0 is the only zero of f inside R.*

PROOF. We begin by observing that

$$\operatorname{Im} f(x + \mathrm{i}y) = (\cos x + x \sin x)\sinh y - y \cos x \cosh y.$$

For $\pi/2 < x \le \pi$ we trivially have $\cos x + x \sin x \ge \cos x$, and so

$$\operatorname{Im} f(x + \mathrm{i}y) \ge -\cos x (y \cosh y - \sinh y)$$

if $y > 0$. Comparing the series $y \cosh y = y + y^3/2! + \cdots$ and $\sinh y = y + y^3/3! + \cdots$ yields $y \cosh y > \sinh y$ for $y > 0$, from which we conclude that $\operatorname{Im} f(x + \mathrm{i}y) > 0$. For $x = \pi/2$ and $y > 0$ we have $\operatorname{Im} f(x + \mathrm{i}y) = \frac{\pi}{2}\sinh y > 0$. Thus, due to $f(-z) = -f(z)$, we know that f has no zeros in the domain $R \setminus ([\pi/2, \pi] \times \{0\})$ but has exactly one zero in $]\pi/2, \pi[\times \{0\}$, since f is strictly increasing there. □

THEOREM 2. *Let z_0 denote the unique zero of the entire function $f(z) = \sin z - z \cos z - \pi/2$ inside the interval $]\pi/2, \pi[$.*

1. *We have*

$$z_0 = \frac{\displaystyle\oint_{|z - 3\pi/8| = \pi/4} z\,dz/(\sin z - z\cos z - \pi/2)}{\displaystyle\oint_{|z - 3\pi/8| = \pi/4} dz/(\sin z - z\cos z - \pi/2)}.$$

2. *In the situation of the goat problem, the radius R of k_2 is given by*

$$R = 2r \cos\left(\frac{1}{2} \frac{\displaystyle\oint_{|z - 3\pi/8| = \pi/4} z\,dz/(\sin z - z\cos z - \pi/2)}{\displaystyle\oint_{|z - 3\pi/8| = \pi/4} dz/(\sin z - z\cos z - \pi/2)}\right).$$

Numerical Approximation

Note that the closed-form expression presented in Theorem 2 gives rise to a numerical approximation of z_0, as observed in [13]. Specifically, in the situation of Theorem 1, assume that the contour C is parameterized by $\gamma(t) = w + r\exp(2\pi \mathrm{i} t)$, $t \in [0, 1]$, for some $w \in \mathbb{C}$, $r > 0$. Then the equation for z_0 can be rewritten as

[4] Recall that the zero set of a nonzero analytic function that is defined on an open connected set is discrete.

$$z_0 = w + r\,\frac{\int_0^1 g(t)\mathrm{e}^{4\pi i t}\,dt}{\int_0^1 g(t)\mathrm{e}^{2\pi i t}\,dt} = w + r\,\frac{c_{-2}}{c_{-1}},$$

where $g(t) = 1/f(w + r\exp 2\pi i t)$, $0 \le t \le 1$, and c_k, $k \in \mathbb{Z}$, denote the Fourier coefficients of g, defined by

$$c_k = \int_0^1 g(t)\mathrm{e}^{-2\pi i k t}\,dt \quad (k \in \mathbb{Z}).$$

An efficient algorithm to compute these coefficients is provided by the fast Fourier transform (FFT), which is based on a sampling vector $(t_0, t_1, \ldots, t_{N-1})$, $0 = t_0 < \cdots < t_{N-1} < 1$, with equidistant elements t_j, where N is a power of 2. The following table shows some approximations (to eleven decimal places) using FFT:

N	z_0	R/r
8	1.90567113225	1.15874852123
32	1.90569572932	1.15872847301
128	1.90569572931	1.15872847302

College of Mathematics and Statistics
Chongqing University
Chongqing
China
e-mail: kubbeldieratz@gmx.de

REFERENCES

[1] E. E. Burniston and C. E. Siewert. The use of Riemann problems in solving a class of transcendental equations. *Proc. Camb. Phil. Soc.* 73:1 (1973), 111–118.

[2] M. Dellnitz, O. Schütze, and Q. Zheng. Locating all the zeros of an analytic function in one complex variable. *J. Comput. Appl. Math.* 138 (2002), 325–330.

[3] M. Fraser. A tale of two goats. *Math. Mag.* 55:4 (1982), 221–227.

[4] M. Fraser. The grazing goat in n dimensions. *College Math. J.* 15:2 (1984), 126–134.

[5] A. Haji-Sheikh and J. V. Beck. An efficient method of computing eigenvalues in heat conduction. *Numerical Heat Transfer Part B: Fundamentals* 38:2 (2000), 133–156.

[6] N. N. Heath. *The Ladies' Diary: or Woman's Almanack, for the Year of Our Lord, 1749*, Vol. 48. A. Wilde, 1749.

[7] H. Heaton, H. W. Holycross, and C. E. Myers. Problem 30. *Amer. Math. Monthly* 1:4 (1894), 395–396.

[8] J. Herlocker and J. Ely. An automatic and guaranteed determination of the number of roots of an analytic function interior to a simple closed curve in the complex plane. *Rel. Comput./ Nadezhn. Vychisl.* 1:3 (1995), 239–249.

[9] M. E. Hoffman. The bull and the silo: an application of curvature. *Amer. Math. Monthly* 105:1 (1998), 55–58.

[10] D. Jackson. Non-essential singularities of functions of several complex variables. *Ann. of Math.* 17:4 (1916), 172–179.

[11] D. Jackson. Roots and singular points of semi-analytic functions. *Ann. of Math.* 19:2 (1917), 142–151.

[12] R. Luck and J. W. Stevens. Exact solutions for transcendental equations. *SIAM Rev.* 44:2 (2002), 227–233.

[13] R. Luck, G. J. Zdaniuk, and H. Cho. An efficient method to find solutions for transcendental equations with several roots. *International Journal of Engineering Mathematics*, Article ID 523043, 4 pp., 2015.

[14] M. D. Meyerson. Return of the grazing goat in n dimensions. *Two-Year Coll. Math. J.* 15:5 (1984), 430–432.

[15] F. de Monte. Multi-layer transient heat conduction using transition time scales. *Int. J. Therm. Sci.* 45:9 (2006), 882–892.

[16] N. I. Muskhelishvili. *Singular Integral Equations*. Dover, 1953.

[17] C. E. Myers, J. W. Watson, et al. Solution to Problem 30. *Amer. Math. Monthly.* 1:11 (1894), 395–396.

[18] A. Neumaier. An existence test for root clusters and multiple roots. *Z. Angew. Math. Mech.* 68:6 (1998), 256–257.

[19] J. W. Stevens and R. Luck. Explicit approximation for all eigenvalues of the 1-D transient heat conduction equations. *Heat Transfer Eng.* 20:2 (1999), 35–41.

[20] J. S. Phelps, III and M. J. Funcheon. Solutions of a transcendental equation basic to the study of composite media heat transfer. *J. Comput. Appl. Math.* 10:2 (1984), 135–139.

[21] G. Pickett and Y. Millev. On the analytic inversion of functions, solution of transcendental equations and infinite self-mappings. *J. Phys. A.* 35:20 (2002), 4485–4494.

[22] J. Rokne. Automatic error bounds for simple zeros of analytic functions. *Comm. ACM.* 16 (1973), 101–104.

[23] N. N. Upnorensis. *The Ladies' Diary: or, Woman's Almanack, for the Year of Our Lord, 1748*, Vol. 45. A. Wilde, 1748.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.