# CHAIN RULE FOR SHIFTS OF CALCS ON CANCELLATIVE MONOIDS

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Let $A, B$ be cancellative monoids. A map $e\colon A \to B$ is *calc* if $e(1) = 1$ and $e(ac) \in e(a)B$ for all $a, c \in A$.

The *shift* of a calc map $e\colon A \to B$ by an element $a \in A$, in symbols $e^a = f$, is the map $f\colon A \to B$ such that $e(ac) = e(a)f(c)$ for all $a, c \in A$. Note the use of cancellative property to establish well-defined $f$.

Let $A, B$ be cancellative monoids, and let $E$ be the set of calc maps $e\colon A \to B$. We can turn $E$ into a right $A$-set as follows: let $e \cdot a = e^a$. The reader can verify that $e^{ac} = (e^a)^c$ and $e^1 = e$ for all $a, c \in A$ and $e \in E$.

We can turn $B \times E$ into a right $A$-set with $(b, e) \cdot a = (be(a), e^a)$.

Now suppose $A = B$. We can define a multiplication on $A \times E$ as $(a, e) \cdot (b, f) = (ae(b), e^b \circ f)$. This turns $A \times E$ into a monoid. For this, we need the

Prop. (Chain rule for shifts of calc maps) Let $A$ be a cancellative monoid and $E$ the set of calc maps with domain and codomain $A$. Then for all $e, f \in E$ and $a \in A$, we have

$$(e \circ f)^a = e^{f(a)} \circ f^a.$$

Proof. Exercise.