Exercise 5.55 (Dual space of a direct sum). Let I be a non-empty set of indices and let $\{V_i\}_{i\in I}$ be vector spaces over a field $\mathbb{F}$. Show

$$\left(\bigoplus_{i\in I} V_i\right)^* \cong \prod_{i\in I} V_i^*,$$

by constructing a canoncial isomorphism.

Proof. When $\{V_i\}_{i\in I}$ are all the 0 vector space, then the dual space of their direct sum and their inner product are the 0 vector space for all $i \in I$; hence, the 0 mapping is a natural isomorphism.
We presuppose that there is vector space a V_i for $i \in I$ that is not the 0 vector space. Consider

$$\Phi : \left(\bigoplus_{i\in I} V_i\right)^* \ni \alpha \mapsto (\alpha_i)_{i\in I} \in \prod_{i\in I} V_i^*$$

where α_i is defined as

$$\alpha_i : \bigoplus_{i\in I} V_i \mapsto (\alpha \circ \iota \circ \mathrm{pr}_i)(v) \in \mathbb{F}$$

for a functional $\alpha \in (\bigoplus_{i\in I} V_i)^*$ and an index $i \in I$; additionally, it is ι is the inclusion into the vector space $\bigoplus_{i\in I} V_i$ and pr_i is the projection onto the i-th component of a vector $v \in \bigoplus_{i\in I} V_i$.
Let α, $\beta \in (\bigoplus_{i\in I} V_i)^*$ and λ, $\mu \in \mathbb{F}$. It follows,

$$\Phi((\lambda\alpha) + (\mu\beta)) = ((\lambda\alpha_i) + (\mu\beta_i))_{i\in I} = \lambda(\alpha_i)_{i\in I} + \mu(\beta_i)_{i\in I} = \lambda\Phi(\alpha) + \mu\Phi(\beta).$$

This shows that Φ is a linear transformation. Let $\alpha \in (\bigoplus_{i\in I} V_i)^*$ and $\alpha \neq 0$. It exists a $v \in \bigoplus_{i\in I} V_i$ with $\alpha(v) \neq 0$; meaning, there exists an $i \in I$, such that $\alpha_i(v) \neq 0$. Therefore, $\Phi(\alpha)(v) = (\alpha_i(v))_{i\in I} \neq 0$ and in consequence $\Phi(\alpha) \neq 0$. Because the Φ is a linear transformation, it is $\Phi(0) = 0$. The injectivity of Φ results from $\ker \Phi = \{0\}$. Let $(\beta_i)_{i\in I} \in \prod_{i\in I} V_i^*$; note that the indices do not refer to the definition of α_i. Let $\{b^i_j\}_{j\in J}$ for $i \in I$ be bases of the respective vector spaces V_i. Define $\gamma \in (\bigoplus_{i\in I} V_i)^*$ as the functional with $\{\gamma(b^i_j) = \beta_i(b^i_j)\}_{j\in J}$ for all $i \in I$; it is $\Phi(\gamma) = (\gamma_i)_{i\in I} = (\beta_i)_{i\in I}$ therefore and Φ is surjective; note that the indices of $(\gamma_i)_{i\in I}$ now do refer to the definition of α_i. We conclude: The mapping Φ is a natural isomorphism from $\bigoplus_{i\in I} V_i$ to $\prod_{i\in I} V_i^*$. □