Exercise 5.56 (Dual bases and matrix representation). Let V and W be finite-dimensional vector spaces and let $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$ be ordered bases. It is $\Phi_{ij} \in \mathbb{F}$ now given by $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Define the mapping

$$\Phi : V \ni v \mapsto \Phi(v) = \sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} a_j^*(v) b_i \in W.$$

Show that Φ is a linear transformation and determine its matrix representation ${}_B[\Phi]_A$.

Proof. Let $v, u \in V$ and $\lambda, \mu \in \mathbb{F}$. Because of the linearity of the functionals a_j^* for $j = 1, \ldots, n$, we have

$$\begin{aligned}
\Phi(\lambda u + \mu v) &= \\
\sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} a_j^*(\lambda u + \mu v) b_i &= \\
\sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} [\lambda a_j^*(u) + \mu a_j^*(v)] b_i &= \\
\sum_{i=1}^{m} \sum_{j=1}^{n} [\Phi_{ij} \lambda a_j^*(u) + \Phi_{ij} \mu a_j^*(v)] b_i &= \\
\sum_{i=1}^{m} \sum_{j=1}^{n} [\Phi_{ij} \lambda a_j^*(u) b_i + \Phi_{ij} \mu a_j^*(v) b_i] &= \\
\sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} \lambda a_j^*(u) b_i + \sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} \mu a_j^*(v) b_i &= \\
\lambda \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} a_j^*(u) b_i \right] + \mu \left[\sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} a_j^*(v) b_i \right] &= \\
\lambda \Phi(u) + \mu \Phi(v).
\end{aligned}$$

This shows that Φ is a linear transformation. □

We now determine the matrix representation ${}_B[\Phi]_A$. Let $k = 1, \ldots, n$. It is

$$\Phi(a_k) = \sum_{i=1}^{m} \sum_{j=1}^{n} \Phi_{ij} a_j^*(a_k) b_i = \sum_{i=1} \Phi_{ik} a^*(a_k) b_i = \sum_{i=1} \Phi_{ik} b_i,$$

because of the definition of the dual basis a_j^* for $j = 1, \ldots, n$. Therefore,

$${}_B[\Phi]_A = \begin{pmatrix} \Phi_{11} & \Phi_{12} & \ldots & \Phi_{1n} \\ \Phi_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \Phi_{m1} & \ldots & \ldots & \Phi_{m,n} \end{pmatrix}.$$