Finding Vector Potentials[1]

Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$. If $\bigtriangledown \cdot \mathbf{F} = 0$ then $\mathbf{F}$ is said to be **divergence free**. For divergence free vector fields it is known that there exists a vector field $\mathbf{G}$ such that $\mathbf{F} = \bigtriangledown \times \mathbf{G}$. Such a vector field is called a **vector potential** for $\mathbf{F}$. Here we shall develop a method for finding a vector potential. Write $\mathbf{F} = \langle F_1, F_2, F_3 \rangle$ and $\mathbf{G} = \langle G_1, G_2, G_3 \rangle$ then solving $\bigtriangledown \times \mathbf{G} = \mathbf{F}$ amounts to simultaneously solving

$$\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z} = F_1,$$

$$\frac{\partial G_1}{\partial z} - \frac{\partial G_3}{\partial x} = F_2,$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3.$$

To do this we exploit the fact that $\mathbf{G}$ is far from unique. Suppose $\mathbf{H} = \langle H_1, H_2, H_3 \rangle$ has curl $\mathbf{F}$. Let f be any differentiable scalar function on $\mathbb{R}^3$. Notice that

$$\bigtriangledown \times (\mathbf{H} + \bigtriangledown f) = \bigtriangledown \times \mathbf{H} + \bigtriangledown \times (\bigtriangledown f) = \bigtriangledown \times \mathbf{H} + \mathbf{0} = \bigtriangledown \times \mathbf{H}.$$

Thus $\mathbf{G} = \mathbf{H} + \bigtriangledown f$ is also a vector potential for $\mathbf{F}$. Choose f so that $\dfrac{\partial f}{\partial z} = -H_3$. This can be done by simply integrating H_3 with respect to z,

$$f(x, y, x) = -\int H_3(x, y, z)\, dz,$$

using any choice you like for the arbitrary constant of integration. Thus we can assume that $\mathbf{G} = \langle G_1, G_2, 0 \rangle$ and we now only need to solve

$$-\frac{\partial G_2}{\partial z} = F_1,$$

$$\frac{\partial G_1}{\partial z} = F_2,$$

$$\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y} = F_3.$$

We illustrate how to do this with two examples.

Example 1. Let $\mathbf{F} = \langle x^2, 3xz^2, -2xz \rangle$. Show that $\bigtriangledown \cdot F = 0$ and find a vector potential $\mathbf{G}$ for $\mathbf{F}$. Verify directly that $\bigtriangledown \times \mathbf{G} = \mathbf{F}$.

Solution. Firstly, $\bigtriangledown \cdot \mathbf{F} = 2x + 0 - 2x = 0$. Next

$$-\frac{\partial G_2}{\partial z} = x^2 \implies G_2(x, y, z) = -x^2 z + C_2(x, y)$$

and

$$\frac{\partial G_1}{\partial z} = 3xz^2 \implies G_1(x, y, z) = xz^3 + C_1(x, y)$$

where the C_1 and C_2 are arbitrary differentiable functions of x and y only. And $(G_2)_x - (G_1)_y = -2xz$ implies

$$\left(-2xz + \frac{\partial C_2}{\partial x}\right) - \left(0 + \frac{\partial C_1}{\partial y}\right) = -2xz$$

or

$$\frac{\partial C_2}{\partial x} = \frac{\partial C_1}{\partial y}.$$

So we just need to pick functions $C_1(x,y)$ and $C_2(x,y)$ that satisfy this condition. We will take the easy way out and just set both equal to zero! Thus our solution is

$$\mathbf{G} = \left\langle xz^3, -x^2z, 0\right\rangle.$$

Finally we check our answer.

$$\triangledown \times \left\langle xz^3, -x^2z, 0\right\rangle = \left\langle 0-(-x^2), 3xz^2 - 0, -3x^2z - 0\right\rangle = \mathbf{F}.$$

□

Example 2. Let $\mathbf{F} = \langle yz, xz, xy\rangle$. Show that $\triangledown \cdot F = 0$ and find a vector potential $\mathbf{G}$ for $\mathbf{F}$. Verify directly that $\triangledown \times \mathbf{G} = \mathbf{F}$.

Solution. Firstly, $\triangledown \cdot \mathbf{F} = 0 + 0 + 0 = 0$. Next we have

$$-\frac{\partial G_2}{\partial z} = yz \implies G_2(x,y,x) = -\frac{1}{2}yz^2 + C_2(x,y),$$
$$\frac{\partial G_1}{\partial z} = xz \implies G_1(x,y,x) = \frac{1}{2}xz^2 + C_1(x,y).$$

Then $(G_2)x - (G_1)_y = xy$ implies $(C_2)_x - (C_1)_y = xy$. We just have to pick functions that do this. Let $C_2 = \frac{1}{2}x^2y$ and $C_1 = 0$. Thus our solution is

$$\mathbf{G} = \left\langle \frac{1}{2}xz^2, \frac{1}{2}x^2y - \frac{1}{2}yz^2, 0\right\rangle.$$

Finally we check our answer.

$$\triangledown \times \left\langle \frac{1}{2}xz^2, \frac{1}{2}x^2y - \frac{1}{2}yz^2, 0\right\rangle = \langle 0-(-yz), xz - 0, xy - 0\rangle = \mathbf{F}.$$

□

In physics courses vector potentials come up because magnetic fields are divergence free and so have vector potentials. However, they typically take a different approach to finding these. They use the fact that usually a magnetic field is the result of a stream of charged particles called a current. Using this one can find a vector potential that is more physically natural.

Given a divergence free vector field $\mathbf{F}$ one might wonder what is the set of all vector potentials for $\mathbf{F}$. If the $\mathbf{G}$ is a vector potential for $\mathbf{F}$ and f is any differentiable scalar function then we have seen that $\mathbf{G} + \triangledown f$ is also a vector potential for $\mathbf{F}$. It can be shown that these are in fact all of the possible vector potentials of $\mathbf{F}$. This is analogous to the fact that if $h'(x) = q(x)$ then the set of all anti-derivatives of $q(x)$ can be expressed as $h(x) + C$. Likewise for all the scalar potentials of a conservative field.

Problems.

(1) Redo Example 1 but assume $G_2 = 0$. Show that the difference between your answer and the one obtained before has gradient zero.

(2) Redo Example 2 but assume $G_1 = 0$. Show that the difference between your answer and the one obtained before has gradient zero.

(3) Let $\mathbf{F} = \langle y, z, x \rangle$. Show that $\mathbf{F}$ is divergence free; find a vector potential; then check your result.

(4) Let $\mathbf{F} = \langle z^2 x - 1, -z^2 y, 1 - x^2 \rangle$. Show that $\mathbf{F}$ is divergence free; find a vector potential; then check your result.

(5) Let $\mathbf{F} = \langle x, 0, 0 \rangle$. Then $\bigtriangledown \cdot \mathbf{F} = 1 \neq 0$. Try to find a vector potential anyway and see what goes wrong.