INTEGER TOPOLOGICAL PROOF OF DIRICHLET'S THEOREM

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Abstract. Closure of product sets in Golomb's topology provides a substantial condition for Dirichlet's theorem on prime numbers in relatively prime arithmetic progressions.

1. Introduction

Arithmetic progressions of the form $a\mathbb{N}+b$ with coprime coefficients contains infinitely many prime numbers as it was proven by Dirichlet back in 1837 [1], we use a few properties of Golomb's topology [2] over the integers $\mathbb{Z}$ by applying the same approach as in Furstenberg's on the infinitude of Primes [3] to provide another proof of Dirichlet's result.

Recall that Golomb's topology takes as a basis the collection of all sets $p\mathbb{Z}+q$ with relatively prime coefficients (p,q), however in the classical definition the topology is based on the positive integers, this is crucial because otherwise it will apear to be a discrete topology since a few basic properties are required in order to confirm our point, which also require it to be a profinite topological group, you may also notice that it is a regular space [4] as might appear from the first property of 2.0.0.1 and later in 2.0.1
Relatively prime arithmetic progression can be expressed analytically as $S(p,q) = p\mathbb{Z}+q,\ \gcd(p,q)=1$ with $q \notin S_0(p,q) = p\mathbb{Z}_0+q$ where we notate $\mathbb{Z}_0 = \mathbb{Z}\setminus\{0\}$, along with that introduce the bijective arithmetic progression $s(n) = pn+q$ and it's image $S(p,q)$ where the set of prime generating numbers is $s_p = s^{-1}\big(S(p,q)\cap\mathbb{P}\big)$ and it's complement $\mathbb{Z}\setminus s_p = s^{-1}\big(S(p,q)\setminus\mathbb{P}\big)$

2. Infinitude of prime numbers in Golomb's topology

Let us recall a few notable properties of Golomb's topology

Lemma 2.0.0.1 (Closure of $S(p,q)$ and finite sets)**.** *Golomb's topology endows the following simple properties.*

(1) *Every relatively prime arithmetic progression $S(p,q)$ is clopen.*
(2) *Any finite set is closed but not open.*

Proof. First property is due to the fact that $S(p,q)$ is the complement of a union of other arithmetic progressions $S\big(p,\mathbb{N}_p\setminus q\big)$, secondly it is obvious that a finite set P cannot be open, it's however closed as it will appear that $S_0(p,0)$ for each $p\in P$ is open, since for any $z\in\mathbb{Z}_0$ we will find α such that $S(\alpha,pz)\subset S_0(p,0)$. $\square$

The following part uses the basic topological properties of the product set $S(p,1)S(p,q)$ in order to show that the set of positive prime numbers $\mathbb{P} \ni 1$ has infinitely many elements to be found in a relatively prime arithmetic progression.

Theorem 2.0.1 (Closure of $S_0(s(n),n)$ under $\mathbb{Z} \setminus s_p$)**.** *There is a closure $s_{cl(n)}$ of a relatively prime arithmetic progression $S_0(s(n),n)$ under $\mathbb{Z} \setminus s_p$ for $c = \max s_p$*

Proof. Assume that there is such closure $s_{cl(n)} \subseteq \mathbb{Z} \setminus s_p$ of $S_0(s(n),n)$ then it must be obvious that $s_{cl(n)} \subseteq S(c, \mathbb{N}_c \setminus s_p) \cup S(a,b)$ since it's clearly disjoint from s_p with some fitting coprime numbers a, b.

Those can be found via the intersection

$$S_0(s(n),n) \cap S(c,s_p) \subseteq S(lcm(c,s(n)), s(n)\mathbb{Z}_h + n)$$

where $h = \frac{lcm(s(n),c)}{s(n)}$ and $\mathbb{Z}_h = \mathbb{Z} \cap [-h,h] \setminus \{0\}$ implying

$$(a,b) = (lcm(c,s(n)), s(n)\mathbb{Z}_h + n)$$

now notice that whenever $lcm(s(n),c) = s(n)$ the index h is necessarily $h = \frac{lcm(s(n),c)}{c}$ and $b = c\mathbb{Z}_h + s_p$ hence $s_{cl(n)}$ must exist. □

There can be found a closure of $S_0(s(n),n)$ inside $\mathbb{Z} \setminus s_p$ which will appear to be closed as opposed to our initial assumption, proving the main result.

Theorem 2.0.2 (Infinitude of primes in arithmetic progressions)**.** *There are infinitely many prime numbers in relatively prime arithmetic progressions.*

Proof. Assume the finitude of prime numbers in $S(p,q)$ implying that the corresponding finite primes generating set s_p cannot be open 2.0.0.1.

The non-prime product set $S_0(p,1)\big(S(p,q) \cap \mathbb{P}\big) \subset S(p,q) \setminus \mathbb{P}$ can be excluded in the following way

$$S_{cl}(p,q) = S(p,q) \setminus \mathbb{P} \setminus S_0(p,1)\big(S(p,q) \cap \mathbb{P}\big)$$

It's possible to show that $s^{-1}(S_{cl}(p,q))$ must be clopen since it's complement is as deduced shortly below

$$\begin{aligned}
\mathbb{Z} \setminus s^{-1}(S_{cl}(p,q)) &= s^{-1}\big(S(p,q) \setminus S_{cl}(p,q)\big) \\
&= s^{-1}\big(S(p,1)\big(S(p,q)\big)\big) \\
&= \bigcup_{s(n)\in\mathbb{P}} S(s(n),n)
\end{aligned}$$

Following the closure $s_{cl(n)}$ of $S_0(s(n),n)$ under $\mathbb{Z} \setminus s_p$ as proven previously 2.0.1 which is clopen for any $s(n) \in \mathbb{P}$ we conclude that $\mathbb{Z} \setminus s_p$ is clopen since it is a finite union of all such $s_{cl(n)}$ and of $s^{-1}(S_{cl}(p,q))$ however that's contradictory to our initial argument of s_p, hence there must be infinitely many prime numbers in $S(p,q)$. □

References

[1] Dirichlet, P. G. L. (1837), "Beweis des Satzes, dass jede unbegrenzte arithmetische Progression, deren erstes Glied und Differenz ganze Zahlen ohne gemeinschaftlichen Factor sind, unendlich viele Primzahlen enthält" [Proof of the theorem that every unbounded arithmetic progression, whose first term and common difference are integers without common factors, contains infinitely many prime numbers], Abhandlungen der Königlichen Preußischen Akademie der Wissenschaften zu Berlin, 48: 45–71

[2] Golomb, S. W. (1959). A Connected Topology for the Integers. The American Mathematical Monthly, 66(8), 663–665. doi:10.2307/2309340

[3] Furstenberg, H. (1955). On the Infinitude of Primes. The American Mathematical Monthly, 62(5), 353–353. doi:10.2307/2307043

[4] Szczuka, Paulina. (2014). Regular open arithmetic progressions in connected topological spaces on the set of positive integers. Glasnik Matematicki. 49. 13-23. doi:10.3336/gm.49.1.02.