Let $a_1 < a_2 < \cdots < a_m$ and $b_1 < b_2 < \dots b_n$ be real numbers such that for any real x:

$$\sum_1^m |a_i - x| = \sum_1^n |b_i - x|$$

Show that $m = n$ and $a_i = b_i$, for all i.

Let:

$$a_0 = b_0 = -\infty \qquad\qquad a_{m+1} = b_{n+1} = +\infty$$

Define k_x, l_x to be functions such that:

$$a_{k_x} \leq x < a_{k_x+1} \qquad\qquad b_{k_x} \leq x < b_{k_x+1}$$

See that it is well defined for any $x \in \mathbb{R}$. Then the equation simplifies to:

$$[(m-n) - 2(k_x - l_x)]x - \sum_1^{k_x} a_i + \sum_{k_x+1}^{m} a_i + \sum_1^{l_x} b_i - \sum_{l_x+1}^{m} b_i = 0 \tag{1}$$

which is a constant, namely 0.

Let $x \in (-\infty, \min\{a_1, b_1\})$. Then, $k_x = l_x = 0$ and (1) reduces to:

$$(m-n)x + \sum_1^m a_i - \sum_1^n b_i = 0 \tag{2}$$

The only way (2) can be constant over the interval is if:

$$m = n$$

Suppose $a_i \neq b_i$ for some i. Let i be the smallest (positive) integer such that $a_i \neq b_i$. Then $a_{i-1} = b_{i-1}$. Assume without loss in generality that $a_i < b_i$, and consider the interval $[a_i, \min\{a_{i+1}, b_i\})$. See that it is non-empty. Over this interval, it must be that:

$$k_x = i \wedge l_x = i - 1 \implies (k_x - l_x) = 1$$

But then (1) can no more be a constant. Hence, $a_i = b_i$ for all i.