Extended Real Intervals and the Topological Closure of Extended Real Relations

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Contents

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Introduction

The set of real numbers is:

$$\mathbb{R} = \{x : -\infty < x < \infty\}. \tag{1}$$

The set of extended real numbers is the real numbers augmented with signed infinities:

$$\mathbb{R}^* = \mathbb{R} \cup -\infty \cup +\infty. \tag{2}$$

The goal of this paper is to develop the *closed* system of interval arithmetic operations and relations on the set of extended intervals with extended real endpoints. No undefined operator-operand combinations can exist in a closed system. Because the results of division by zero and indeterminate forms[1] are not single points neither

[1] An indeterminate form is an expression such as $(+\infty) - (+\infty)$, $\frac{0}{0}$, $\frac{\pm\infty}{\pm\infty}$, $0 \times (\pm\infty)$, $(\pm\infty)^0$, 0^0, or $(\pm 1)^{(\pm\infty)}$, for which there is no single defined value. An indeterminate form may result from replacing the limit of a composite function, such as $\lim_{x \to 0} f(g(x), h(x))$, by the composite of its limits, $f\left(\lim_{x \to 0} g(x), \lim_{x \to 0} h(x)\right)$. For example: $\lim_{x \to 0} \frac{x}{x} = 1$, but $\frac{0}{0}$ is indeterminant.

$(+\infty) - (+\infty)$	$(+\infty) \times 0$	$(+\infty) \div (+\infty)$
$(-\infty) + (+\infty)$	$(-\infty) \times 0$	$(-\infty) \div (+\infty)$
$(+\infty) + (-\infty)$	$0 \times (+\infty)$	$(+\infty) \div (-\infty)$
$(-\infty) - (-\infty)$	$0 \times (-\infty)$	$(-\infty) \div (-\infty)$

TABLE 1 Indeterminate arithmetic forms.

the real, nor the extended real number systems are closed. For example, in addition to division by zero, the operand-operator combinations in Table 1 are undefined.

Closed *interval* systems exist because intervals are not single points, but compact sets of points. In the title of "*Interval Arithmetic as a Closed Arithmetic System on a Computer*," [**?**], Hanson implies a closed interval system is defined. However, division by zero and the indeterminate forms in Table 1 are specifically excluded. In "*A More Complete Interval Arithmetic*," [**?**], Kahan defines a closed interval system, including division by intervals containing zero. However, a detailed justification of the proposed definitions has not been developed and the Kahan system produces intervals that are neither as narrow as possible nor as convenient to represent as the intervals in other possible closed interval systems. For example, the decision to include open intervals leads to an internal machine representation requiring extra bits in addition to a pair of floating-point values. Nevertheless, it is a significant achievement that Kahan recognized exterior intervals[2] can be used to sharply[3] bound the set of values resulting from division by intervals containing zero.

Interval Analysis Overview from a Mathematical Perspective

This section contains an informal overview of the main points in the development. After analysis and relation preliminaries in Sections and , Section defines notation and terms used in the remainder of the paper.

The central problem of interval analysis is to bound the set of results from the point evaluation of an expression at every value in the argument intervals. A real expres-

[2] An *exterior interval* is the union of two semi-infinite intervals, as in $[-\infty, a] \cup [b, +\infty]$ with $a < b$.
[3] An interval bound is *sharp* if it is as narrow as possible and still a bound.

sion of n singleton set arguments, $\{x_i\}$, is denoted: $f(\{\mathbf{x}\}) = f(\{x_1\}, \ldots, \{x_n\})$. A singleton set has exactly one member. The arguments of expressions that produce set-valued results are sets.

The components of the expressions of interest when evaluated at a particular point, $\mathbf{x}_0$ are:

- argument values x_{0i} of the real variables, x_i;
- the basic arithmetic operations (BAOs) $+$, $-$, $\times$, and $\div$;
- constants; and
- other functions or relations.

An interval vector $[\mathbf{X}] = ([X_1], \ldots, [X_n])$ is simply a vector of real intervals, $[X_i]$. All intervals are enclosed in brackets to distinguish them from sets, which are represented using unbracketed, uppercase letters. In this paper, $[X]$ is an interval and X is a set that may or may not be an interval. Braces always surround singleton sets to distinguish them from intervals, which are enclosed in brackets, and points, which are not enclosed in braces or in brackets.

When evaluated over a set, $\mathbf{X}_0$, an expression is simply the union of expression values at every point in the range set:

$$f(\mathbf{X}_0) = \left\{ z \;\middle|\; \begin{array}{l} z \in f(\{\mathbf{x}_0\}) \\ \mathbf{x}_0 \in \mathbf{X}_0 \end{array} \right\}. \tag{3}$$

The central problem of interval analysis is solved with an enclosure, $f([\mathbf{X}_0])$, for the range set, $f(\mathbf{X}_0)$ of f over the interval $\mathbf{X}_0$:

$$f([\mathbf{X}_0]) \supseteq \mathrm{hull}(f(\mathbf{X}_0)) \tag{4}$$

given the interval hull of the set, R, is

$$\mathrm{hull}(R) = \left[\inf(R), \sup(R)\right], \tag{5}$$

and

$$[\mathbf{X}_0] \subseteq \mathcal{D}_f, \tag{6}$$

the *natural domain* of f. An expression's domain is *natural* if it is the intersection of operator and intrinsic function domains in the given expression and is formally defined in Section , item 3 on page 9. In (3), as is customary, the n-tuple $[\mathbf{X}]$ is identified with the Cartesian product $[X_1] \otimes \cdots \otimes [X_n]$ (a box in n-space), so that $\mathbf{x} \in [\mathbf{X}]$ is a simple way of writing: $x_i \in [X_i]$ for each i.

Provided $[\mathbf{X}_0]$ is a subset of $\mathcal{D}_f$, the required enclosure is produced by the *interval evaluation* of f at $[\mathbf{X}_0]$. The resulting interval may be an inaccurate approximation of hull $(f(\mathbf{X}_0))$, but is a guaranteed enclosure. In the interval literature, $f([\mathbf{X}])$ is usually written $f(\mathbf{X})$. This practice is natural in the context of overloading the meaning of f to operate on interval instead of real data items. Because this paper is primarily mathematical, $f(\mathbf{X})$ represents the range set, as is customary in mathematics.

From a mathematical perspective, the goal of this paper is to extend the meaning of $f(\{\mathbf{x}\})$ to points outside $\mathcal{D}_f$, in such a way that the process of interval evaluation continues to give valid enclosures when the process is defined as the interval hull of f at the point, $\mathbf{x}_0$. In practice, extension is necessary because interval arguments are not always confined to $\mathcal{D}_f$.

The first step is to precisely define all the needed analysis and relation preliminaries, notation, and terminology. Care with this step is required because points, sets, and intervals are all used in the development. Without clear notation, opportunities for ambiguity will make the exposition cumbersome at best, and unclear at worst.

The key is the introduction of the concept of the *containment set*[4] of f at $\mathbf{x}_0$: the minimum set of values that $f([\mathbf{x}_0])$ must contain, whether or not $\mathbf{x}_0 \subseteq \mathcal{D}_f$. In other words, the interval valuation of f must unconditionally succeed and yield an enclosure of f's containment set.

Development begins in Section with a specification of the properties that the containment set of an expression must satisfy. From these properties, the following main results flow:

1. The *containment constraint* that containment sets must satisfy is defined in Section .
2. The *containment set* is defined in Section .
3. The containment-set closure identity is proved in Section **??**, Theorem 1.
4. Containment sets of basic arithmetic operations are derived in Section .

[4] For the defining properties of the containment set of a relation, see Section on page 18.

5. Containment sets of the ln, exp, and exp $(y \ln x)$ functions are derived in Section .

6. The distinction between variable and value equality is elaborated in Section .

7. Containment-set-equivalent expressions are introduced in Section .

The $I\!R^*$ system is the mathematical foundation for closed interval systems in general, and in particular, the interval system described in "*The 'Simple' Closed Interval System,*" [**?**], operationally defined in "*Implementing the 'Simple' Closed Interval System,*" [**?**], and implemented in "*Forte*™ *Developer 6 Fortran 95,*" [**?**]. Because this system is closed, floating-point runtime exceptions are logically impossible. Hereafter, the "Simple" Closed Interval System is referred to as the Simple System.

The development applies standard mathematical principles outlined in Sections and . The individual mathematical principles are well known, but they have never been combined with the specific aim of constructing closed interval systems.

The term "closed" is used in two ways:

- The system consisting of a set of members and binary operations is closed if any binary operations in the system on members of the set produces another member of the set.
- A topologically-closed set or closed interval contains all the accumulation points in the set or interval.

Analysis Preliminaries

The needed basic notions from analysis are reviewed here. An infinite *sequence* of points $x_1, x_2, x_3, \ldots$ is denoted by parentheses, (x_i). A *subsequence* of (x_i) means a sequence (x_{i_j}) where $i_1, i_2, i_3, \ldots$ is a strictly increasing sequence of indices.

A *metric space* is a set S on which a real-valued distance function $d(x, y) \geq 0$ is defined, such that given $x, y, z \in S$, the following three laws are satisfied:

1. $d\,(x, y) = 0$ iff, $x = y$;

2. $d\,(x, y) = d\,(y, x)$; and,

3. $d(x, y) + d(y, z) \geq d(x, z)$.

The function $d(x, y)$ is said to be a metric for S.

By definition, a sequence (x_i) in S converges to $x \in S$ (equivalently, x is the limit (in the usual sense) of (x_i)), iff

$$d(x_i, x) \to 0 \text{ in } \mathbb{R}. \tag{7}$$

That is, $\lim_{i\to\infty} x_i = x$, or $x_i \to x$, are equivalent ways to write (7).

A subset X of S is a *neighborhood* of a point $a \in S$ iff for some $\epsilon > 0$, X contains the 'ϵ-ball' $\{x \in S \mid d(x, a) < \epsilon\}$ around a. A point x is an *accumulation point* of a subset X of S iff it is the limit of some sequence of points in X; equivalently, if every neighborhood of x meets (has nonempty intersection with) X. A set is *closed* iff it contains all its accumulation points. A set is *open* iff it is a neighborhood of each of its points.

The *closure* of X, denoted by the customary notation $\overline{X}$, consists of all accumulation points of X. The closure of X contains X, since every x is the limit of the sequence in which all elements equal x. The *interior* of X is the set of points of which X is a neighborhood. The *boundary* of X is the set of points that are in X's closure and not in its interior. Basic, and not quite trivial, facts are that a set's:

- closure is closed,
- interior is open,
- boundary is the points common to its closure and the closure of its complement, and
- complement in S is open iff S is closed.

Example 1 In $\mathbb{R}$, let X be the union of $\{1, 1/2, 1/3, 1/4, \ldots\}$ and the open interval $(-1, 0)$. Then the set of accumulation points of X, i.e. its closure, is $X \cup \{-1, 0\}$. The interior is the open interval $(-1, 0)$. The boundary is $\{1, 1/2, 1/3, 1/4, \ldots\} \cup \{-1, 0\}$.

A space in which a definition has been made of exactly which sequences are convergent is called *metrizable* iff there is a metric on it that generates this same meaning of convergence.

The *Cartesian product* $X \otimes Y$ of metric spaces X, Y consists of all ordered pairs (x, y) with $x \in X$ and $y \in Y$. The definition of convergence is that $(x_i, y_i) \to$

(x, y) iff $x_i \to x$ and $y_i \to y$. Convergence and a metric in the product $X_1 \otimes \ldots \otimes X_n$ of n metric spaces are defined by the obvious extension of these constructions.

An essential role is played by the notion of *compactness*. A subset X of a metric space S is *compact* iff every sequence of points in X has a subsequence that converges to a limit that itself is in X. When this is applied to the whole space S, one can say more simply: S is compact iff every sequence in S has a convergent subsequence.

The Cartesian product of compact spaces is compact.

The spaces of primary interest here are $\mathbb{R}$, $\mathbb{R}^*$, and Cartesian products of them. The usual metric on $\mathbb{R}$ is $d(x, y) = |x - y|$. A metric on $\mathbb{R}^*$ is defined below. A key property is that *every closed bounded subset of* $\mathbb{R}$ *is compact*. This property distinguishes $\mathbb{R}$ from 'thinner' sets of numbers like the rationals, and is equivalent to the property that every bounded subset has a least upper bound. Either of these properties may be taken as the foundation of real analysis.

Convergence in the extended reals, $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, +\infty\}$, is defined as follows. The sequence (x_i) converges to a finite x iff the x_i are all finite from some $i = i_0$ onward, and converge to x in the usual way. The sequence (x_i) converges to $+\infty$ iff for any real c no matter how large, there exists an i_0 such that $x_i \geq c$ for all $i \geq i_0$. Convergence to $-\infty$ is defined similarly. This makes $\mathbb{R}^*$ *topologically equivalent* to the closed interval $I = [-1, 1]$, in the sense that there is a map $\phi : \mathbb{R}^* \to I$ such that $x_i \to x$ in $\mathbb{R}^*$ iff $\phi(x_i) \to \phi(x)$ in I. For instance, ϕ can be the function

$$\phi(x) = \begin{cases} -1 & \text{if } x = -\infty, \\ \tanh x & \text{if } x \text{ is finite,} \\ 1 & \text{if } x = +\infty. \end{cases} \tag{8}$$

This equivalence proves $\mathbb{R}^*$ is metrizable, a possible metric being $d(x, y) = |\phi(x) - \phi(y)|$ for $x, y \in \mathbb{R}^*$. Since I is compact, it follows that $\mathbb{R}^*$ is compact. Hence also, $(\mathbb{R}^*)^n$ is metrizable and compact, a possible metric on it being $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |\phi(x_i) - \phi(y_i)|$ for any $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ in $(\mathbb{R}^*)^n$.

Relation Preliminaries

Following standard, set-theory notation:

1. A relation between the sets S (the *source*) and T (the *target*) is by definition a subset F of the Cartesian product $S \otimes T$. If F is such a relation and if X is a

subset of S, the notation $f(X)$ means the set

$$f(X) = \{y \in T \mid \text{there exists } x \in X \text{ such that } (x, y) \in F\}. \tag{9}$$

Note that $f(X)$ is *always* defined but may be the empty set.

2. The *domain* $\mathcal{D}_f$ of F is the set of $x \in S$ such that there is at least one y with $(x, y) \in F$, equivalently such that $f(\{x\})$ is nonempty.

3. The *range* $\mathcal{R}_f$ of F is the set of $y \in T$ such that there is at least one x with $(x, y) \in F$. That is, $\mathcal{R}_f = f\left(\mathcal{D}_f\right)$.

4. When $x \in S$ is such that $f(\{x\})$ is a singleton set $\{y\}$, f is said to be *a function at* x. Then, and only then, is the function notation $y = f(x)$ used. That is, $f(x)$ is the member of $f(\{x\})$ when this is unique, and undefined otherwise. f is said to be *a function* if it is a function at each point of its domain. Thus, if f is a function,

$$f(\{x\}) = \begin{cases} \text{the singleton set } \{f(x)\}, \text{ if } x \in \mathcal{D}_f, \text{ and} \\ \text{the empty set, otherwise.} \end{cases} \tag{10}$$

5. Given another relation $G \subseteq T \otimes U$, then the *composition* or *composite relation* $G \circ F$ is the relation between S and U defined by

$$G \circ F = \left\{ (x, z) \in S \otimes U \;\middle|\; \begin{array}{l} \text{there exists } y \in T \text{ such that} \\ (x, y) \in F \text{ and } (y, z) \in G \end{array} \right\}. \tag{11}$$

This corresponds to the usual meaning of a composition of functions. Namely, if f and g are functions, then $G \circ F$ is the composite function $g(f(x))$ wherever it is defined.

Notation and Terminology

The analysis of containment sets and topological closures includes numerous opportunities for notation ambiguity. The notation described in this section is strictly enforced to distinguish between: variables and particular values they can take on; between points, sets, and intervals; and between functions and relations.

1. The subscript, 0, is used to denote a specific value of a variable. For example, $x_0 = y_0$ denotes the specific values, x_0 and y_0 (of the variables x and y) are the same. The fact that $x_0 = y_0$ does not necessarily imply the variables x and y, or

the expressions they represent, are identical. Two expressions are identical if they have the same domain and the same values for all arguments in their common domain. Therefore, equality of x_0 and y_0 does not necessarily imply the variables x and y can be interchanged. However, if the variables, x and y are identically equal, they are also interchangeable. The consequence of the distinction between *value* and *variable* or *expression equality* is seen in subsequences used to define accumulation points and in interval expressions. For example, if $x_j = y_j$ for all j, the sequences $\left(x_j\right)$ and $\left(y_j\right)$ are identical and have the same limits. If only their limits are equal, the sequences can nevertheless be different. Two identically equal intervals are dependent and have the same value. Two intervals that only have the same value need not be dependent.

2. Customary notation in both the mathematical and interval literature uses uppercase letters to denote sets and intervals, respectively. For clarity when working with both sets and intervals, it is necessary to distinguish between them. Because this paper is primarily concerned with sets rather than intervals, unbracketed uppercase letters represent sets and all intervals are enclosed in brackets. That is, $[X]$ is the closed interval, $\left[\underline{x}, \overline{x}\right] = \left\{z \mid \underline{x} \le z \le \overline{x}\right\}$. The set X is not necessarily an interval, although it can be. Singleton sets are lower-case and enclosed in braces to distinguish them from points, which are neither enclosed in brackets nor braces. The singleton set, $\{x\}$, has only one element, the point, x.

3. The evaluation of an *expression* is any computation defined by the execution of a *code list* (or Wengert list). The following are important to distinguish:

   a a segment of computer code,

   b the expression defined when the code is executed, and

   c the relation or function defined by this expression.

   Irrespective of whether the code includes branches, loops and subprogram calls, array references, or overwriting of a variable's value by a new value, any particular execution is a finite code list of operations where each new computed value is given a different name:

$$\begin{array}{l} \text{Input } x_1, \ldots, x_n. \\ \text{Compute } x_i = e_i(\text{earlier } x_j), \text{ for } i = n+1, \ldots, q+m. \\ \text{Output } x_{q+1}, \ldots, x_{q+m}. \end{array} \tag{12}$$

There are n inputs, $(q-n)$ intermediate variables, and m outputs. Each e_i represents one of the four basic arithmetic operations (BAOs) or some other 'intrinsic' function. Constants may be treated as zero-argument intrinsic functions.

Only the case $m = 1$, a scalar function of several variables, is treated in this paper. By successive substitution, the intermediate variables can be eliminated to give an expression f for the output, which is uniquely determined by the code list:

$$x_{q+1} = f(x_1, \dots, x_n).$$

For instance, if the input is (x_1, x_2), and if the code list is

$$x_3 = x_1 + x_2 \tag{13a}$$

$$x_4 = x_2/x_3 \tag{13b}$$

$$x_5 = x_4 + x_3, \tag{13c}$$

where the output is x_5. Then

$$f(x_1, x_2) = \frac{x_2}{x_1 + x_2} + (x_1 + x_2). \tag{14}$$

The expression must be sufficiently parenthesized to make the order of evaluation clear. Note that information is lost about 'common subexpressions' like $(x_1 + x_2)$ above, but this is irrelevant for the mathematics that follows. The function value $f(\mathbf{x}_0)$ exists at $\mathbf{x}_0 = (x_{01}, \dots, x_{0n})$ in $I\!R^n$ iff, for each 'compute' step in the code list in (12), the arguments to each basic operation e_i lie in the domain of e_i. The set $\mathcal{D}_f$ of such $\mathbf{x}_0$ is called f's *natural domain* provided operator's and intrinsic function's domains are used to define the domain of f in $I\!R^n$. If only the four BAOs are used, then every f defines a *rational function* of the inputs. In this case, $\mathcal{D}_f$ is the set of points $\mathbf{x} \in I\!R^n$, for which divide-by-zero does not occur while evaluating f at $\mathbf{x}$.

4. The point, x, is the single member of the degenerate interval, $[x, x]$ (or equivalently $[x]$), and is also the single member of the singleton set $\{x\}$. Brackets and braces establish context for the interpretation of symbols used to represent expressions. For example, when evaluated either at the point x_0, at the singleton set $\{x_0\}$, or at the degenerate interval $[x_0]$, the expression f is represented:

   a $f(x_0)$ is the function, f, evaluated at the point, $x_0 \in \mathcal{D}_f$, where $\mathcal{D}_f$ is the natural domain of f, b $f(\{x_0\})$ is the relation, f, evaluated at the singleton set $\left\{x_0 \in I\!R^*\right\}$, and

c $f([x_0])$ (or equivalently $f([x_0, x_0])$), is the interval evaluation of the expression, f, at the degenerate interval $\left[x_0 \in I\!R^*\right]$.

Because a code list evaluation can yield a single value or multiple values, the neutral term *expression* is used to refer to the object of a code list evaluation. The present development extends the mathematical foundation under interval arithmetic by using the set-theoretic properties of intervals to define bounds on expressions (whether functions or relations) for any arguments in $I\!R^*$.

5. Bold letters are used to represent vectors of points, sets, and intervals. In particular,

$$\mathbf{x} = (x_1, \cdots, x_n), \tag{15a}$$

$$\{\mathbf{x}\} = \{(x_1, \cdots, x_n)\}, \tag{15b}$$

$$\mathbf{X} = (X_1, \cdots, X_n), \text{ and} \tag{15c}$$

$$[\mathbf{X}] = ([X_1], \cdots, [X_n]) \tag{15d}$$

are respectively:

a a point in n-dimensional Euclidean space, $I\!R^n$,

b a singleton set, the only element of which is a point $\mathbf{x} \in (I\!R^*)^n$,

c the Cartesian product[5], $X_1 \otimes \cdots \otimes X_n \in (I\!R^*)^n$, of sets, X_i, and

d an n-dimensional box, $[X_1] \otimes \cdots \otimes [X_n] \in (I\!R^*)^n$.

From (15b) and the identification of the n-tuple, $\mathbf{X}$, with a Cartesian product in (15c),

$$\{\mathbf{x}\} = (\{x_1\}, \cdots, \{x_n\}). \tag{16}$$

6. Including an argument in the symbolic representation of an expression (for example, $f(\{(x_1, x_2)\})$) implies that both x_1 and x_2 appear in the defining expression for f. For example, all the elements of the vector, $\mathbf{x}$, appear in the defining expression for $f(\{\mathbf{x}\})$.

[5] The n-tuple $\mathbf{X} = (X_1, \cdots, X_n)$ is treated the same as the Cartesian product $X_1 \otimes \cdots \otimes X_n$, because they carry the same information except when any of the X_i are empty.

7. As with points, expressions of point, interval, and set vectors mean different things. That is, $f(\mathbf{x})$, $f(\{\mathbf{x}\})$, and $f([\mathbf{x}])$, are respectively:

   a the function, f, evaluated at the point $\mathbf{x} \in \mathcal{D}_f \in (\mathbb{R})^n$,

   b the relation, f, evaluated at the singleton set $\{\mathbf{x}\}$, given the point, $\mathbf{x} \in (\mathbb{R}^*)^n$, and

   c the interval evaluation of the expression, f, at the degenerate interval $[\mathbf{x}]$.

   In the interval literature, the definition of $f([\mathbf{x}])$ is traditionally limited to single-valued functions with domain $\mathcal{D}_f \in (\mathbb{R})^n$. Although complex intervals are also considered in the interval literature, complex variables and intervals are not considered in this paper.

8. Because $\mathbf{X}$ denotes a set of points in n-dimensional space, the notation $f(\mathbf{X})$ denotes the relation, f, evaluated over all singleton sets, $\{\mathbf{x}\} \in \mathbf{X}$, that is:

$$f(\mathbf{X}) = \left\{ z \,\middle|\, \begin{array}{l} z \in f(\{\mathbf{x}\}) \\ \mathbf{x} \in \mathbf{X} \end{array} \right\}. \tag{17}$$

9. Because $[\mathbf{X}]$ is a box in n-dimensional space, if $[\mathbf{X}] \subseteq \mathcal{D}_f$, the notation $f([\mathbf{X}])$ denotes an interval that must be an enclosure of

$$\left\{ z \,\middle|\, \begin{array}{l} z \in f(\{\mathbf{x}\}) \\ \mathbf{x} \in [\mathbf{X}] \end{array} \right\}. \tag{18}$$

   The question to be answered is: What is the containment set of values that $f([\mathbf{X}])$ must enclose, if $[\mathbf{X}] \nsubseteq \mathcal{D}_f$?

Interval Arithmetic Preliminaries

1. The BAOs in the $\mathbb{R}^*$ system are relations. Evaluating an expression is always on set-valued arguments producing a set-valued result. Normal function evaluation occurs when evaluation at a singleton set $\{\mathbf{x}_0\}$ produces a singleton $\{y_0\}$. Then and only then is the notation $y_0 = f(x_0)$ used.

2. Each extended basic arithmetic operation is the topological closure in $(\mathbb{R}^*)^3$, of (the graph of) the corresponding operation regarded as a subset of $\mathbb{R}^3$.

3. The containment set of a relation evaluated at a point can be disconnected compact sets, not necessarily a single interval. To make an implementable system, the easily described family of extended closed intervals, $\mathbb{IR}^*$ is used. As it must be, the whole of $\mathbb{R}^*$ is in $\mathbb{IR}^*$. Whenever an expression is evaluated, the resulting containment set is replaced by the latter's interval-hull, resulting in an unsharp, but more easily manipulated enclosure.

   A machine-implementable interval arithmetic is obtained if $\mathbb{IR}^*$ comprises all closed intervals with IEEE floating-point-representable endpoints (including $\pm\infty$).

Tables 2 through 5 on page 15 display containment sets for the four BAOs. The notation used to represent a BAO's containment set is:

$$\text{cset}\,(x \text{ op } y, \{(x_0, y_0)\}), \tag{19}$$

where

- cset is the containment-set relation,
- op is one of the BAOs, and
- $\{(x_0, y_0)\}$ is the singleton set, the single member of which is the point (x_0, y_0) at which the containment set is evaluated.

The general form of the containment set of the expression, f, evaluated at the point, $\mathbf{x}_0$, is:

$$\text{cset}\,(f, \{\mathbf{x}_0\}). \tag{20}$$

Using the results developed in Section , these and other containment sets are derived starting in Section . Some explanations are needed regarding the tables of containment sets: All inputs are shown as singleton sets and results are shown either as sets or intervals. To avoid ambiguity, the following customary point arithmetic notation is *not* used:

$$(-\infty) + (-\infty) = -\infty, \tag{21a}$$

$$(-\infty) + y = -\infty, \text{ if } y < +\infty, \text{ and} \tag{21b}$$

$$(-\infty) + (+\infty) = \mathbb{R}^*. \tag{21c}$$

Instead, the tables show results for singleton set inputs to each operation. As seen in equation (17), expression values for a general set input in the $\mathbb{R}^*$ system are simply the union of single-point-argument expression values as single-point arguments range over input sets.

In "*Interval Arithmetic: From Principles to Implementation*," [**?**], Hickey, Ju, and van Emden give an alternative mathematical model and guidelines for implementation. Their model is similar in many ways, but their Principle 1 requires variables to range only over $\mathbb{R}$, so that an interval is defined to be a closed, connected, possibly-empty subset of $\mathbb{R}$. Thus, their intervals have the form $[I] \cap \mathbb{R}$ where $[I]$ is a compact $\mathbb{R}^*$-interval in the $\mathbb{R}^*$ model. Infinities are a notational device for denoting intervals, not points in the number system. In the resulting interval arithmetic, $\{1/0\}$ is the empty set instead of the set $\{-\infty, +\infty\}$. As a consequence, evaluating the expression $\{1/(1 + 1/x)\}$ when $x = 0$, results in containment failures.

In "*New Computer Methods for Global Optimization*," [**?**], Ratschek and Rokne define a two-point compactification of the extended real numbers to permit the starting box in the interval global optimization algorithm to be unbounded. However, they leave undefined both expressions at singular points and indeterminate forms.

Expressions and Functions

The words 'variable' and 'argument' as used herein mean a quantity that can take on a scalar value or a set of values in $\mathbb{R}$ or its extension, $\mathbb{R}^*$. A quantity whose value is a (set of) vectors, i.e. a subset of $\mathbb{R}^n$ or $(\mathbb{R}^*)^n$, is a vector variable or argument. If $X_1, \ldots, X_n$ are sets of scalars and f is a relation of n arguments, the notations $f(X_1, \ldots, X_n)$ and $f(\mathbf{X})$ are used interchangeably, where $\mathbf{X} = X_1 \otimes \cdots \otimes X_n$ is the Cartesian product of the X_i. $\mathbf{X}_0$ is the set, $X_{01} \otimes \cdots \otimes X_{0n}$, of particular values of the arguments, X_i.

Expressions that are *algebraically equivalent*, that is equivalent according to the rules of high school algebra, are not necessarily the same function if, for example, they have different domains. The following are different real functions of real variables. However, for arguments in the intersection of their domains of definition, they produce the same values.

$$f_1(x_1, x_2) = x_2/(x_1 + x_2) \tag{22a}$$

$$f_2(x_1, x_2) = 1 - 1/(1 + x_2/x_1) \tag{22b}$$

$$f_3(x_1, x_2) = 1/(1 + x_1/x_2) \tag{22c}$$

cset $(x + y, \{(x_0, y_0)\})$	$\{-\infty\}$	{real: y_0}	$\{+\infty\}$
$\{-\infty\}$	$\{-\infty\}$	$\{-\infty\}$	$I\!R^*$
{real: x_0}	$\{-\infty\}$	$\{x_0 + y_0\}$	$\{+\infty\}$
$\{+\infty\}$	$I\!R^*$	$\{+\infty\}$	$\{+\infty\}$

TABLE 2 Containment set for addition: cset $(x + y, \{(x_0, y_0)\})$.

cset $(x - y, \{(x_0, y_0)\})$	$\{-\infty\}$	{real: y_0}	$\{+\infty\}$
$\{-\infty\}$	$I\!R^*$	$\{-\infty\}$	$\{-\infty\}$
{real: x_0}	$\{+\infty\}$	$\{x_0 - y_0\}$	$\{-\infty\}$
$\{+\infty\}$	$\{+\infty\}$	$\{+\infty\}$	$I\!R^*$

TABLE 3 Containment set for subtraction: cset $(x - y, \{(x_0, y_0)\})$.

In general, an expression f is different from the function on $I\!R^n$ that it defines, although they are generally given the same name.

Expression Closures

The definition of an expression's closure is:

Definition 1 The closure of the expression f, evaluated at the point $\mathbf{x}_0$ is denoted

cset $(x \times y, \{(x_0, y_0)\})$	$\{-\infty\}$	{real: $y_0 < 0$}	$\{0\}$	{real: $y_0 > 0$}	$\{+\infty\}$
$\{-\infty\}$	$\{+\infty\}$	$\{+\infty\}$	$I\!R^*$	$\{-\infty\}$	$\{-\infty\}$
{real: $x_0 < 0$}	$\{+\infty\}$	$\{x \times y\}$	$\{0\}$	$\{x \times y\}$	$\{-\infty\}$
$\{0\}$	$I\!R^*$	$\{0\}$	$\{0\}$	$\{0\}$	$I\!R^*$
{real: $x_0 > 0$}	$\{-\infty\}$	$\{x \times y\}$	$\{0\}$	$\{x \times y\}$	$\{+\infty\}$
$\{+\infty\}$	$\{-\infty\}$	$\{-\infty\}$	$I\!R^*$	$\{+\infty\}$	$\{+\infty\}$

TABLE 4 Containment set for multiplication: cset $(x \times y, \{(x_0, y_0)\})$.

cset $(x \div y, \{(x_0, y_0)\})$	$\{-\infty\}$	$\{\text{real: } y_0 < 0\}$	$\{0\}$	$\{\text{real:} y_0 > 0\}$	$\{+\infty\}$
$\{-\infty\}$	$[0, +\infty]$	$\{+\infty\}$	$\{-\infty, +\infty\}$	$\{-\infty\}$	$[-\infty, 0]$
$\{\text{real: } x_0 \neq 0\}$	$\{0\}$	$\{x \div y\}$	$\{-\infty, +\infty\}$	$\{x \div y\}$	$\{0\}$
$\{0\}$	$\{0\}$	$\{0\}$	$I\!R^*$	$\{0\}$	$\{0\}$
$\{+\infty\}$	$[-\infty, 0]$	$\{-\infty\}$	$\{-\infty, +\infty\}$	$\{+\infty\}$	$[0, +\infty]$

TABLE 5 Containment set for division: cset $(x \div y, \{(x_0, y_0)\})$.

$\overline{f}\,(\{\mathbf{x}_0\})$, and is defined: if $\mathbf{x}_0 \in \overline{\mathcal{D}}_f$, then

$$\overline{f}\,(\{\mathbf{x}_0\}) = \left\{ z \;\middle|\; \begin{array}{l} z \in \lim_{j\to\infty} y_j \\ y_j \in f\left(\left\{\mathbf{x}_j\right\}\right) \\ \mathbf{x}_j \in \mathcal{D}_f \\ \lim_{j\to\infty}\mathbf{x}_j = \mathbf{x}_0 \end{array} \right\}. \tag{23a}$$

Otherwise, if $\mathbf{x}_0 \notin \overline{\mathcal{D}}_f$, then

$$\overline{f}\,(\{\mathbf{x}_0\}) = \emptyset. \tag{23b}$$

The closure of f is always defined, but may be the empty set. The domain of $\overline{f}$ is the set of argument values for which $\overline{f}\,(\{\mathbf{x}_0\}) \neq \emptyset$, which is the closure of the domain of f, $\overline{\mathcal{D}}_f$. That is,

$$\mathcal{D}_{\overline{f}} = \overline{\mathcal{D}}_f. \tag{24}$$

Given the conditions on the right-hand side of (23a) are satisfied, the closure of an expression is the set of all possible accumulation points in the subsequences whose members are elements of the sets, $f\left(\left\{\mathbf{x}_j\right\}\right)$. If $\mathbf{x}_0 \in \overline{\mathcal{D}}_f$, all subsequences of the sequence $\left(\mathbf{x}_j\right)$, have the common accumulation point, $\mathbf{x}_0$. If $\mathbf{x}_0 \notin \overline{\mathcal{D}}_f$, then $\overline{f}\,(\{\mathbf{x}_0\}) = \emptyset$. Definition 1 imposes no restrictions on the point $\mathbf{x}_0$.

Interval Expressions

When interval arithmetic is used to evaluate an expression at the degenerate interval $[\mathbf{x}] = [(x_1, \cdots, x_n)]$, the code list in (12) becomes:

Input $[x_1], \ldots, [x_n]$.

Compute $[X_i] = e_i\left(\left[x_j\right] \text{ and/or earlier } [X_k]\right)$, for $\begin{cases} i = n+1, \ldots, q+m \\ 1 \leq j \leq n \\ n+1 \leq k \leq q. \end{cases}$

Output $\left[X_{q+1}\right], \ldots, \left[X_{q+m}\right]$.

(25)

Each e_i in (25) represents one of the four basic *interval* arithmetic operations (BIAOs) or some other 'intrinsic' interval function. The interval expression value $f([\mathbf{x}])$ exists at $[\mathbf{x}] = ([x_1], \ldots, [x_n])$ in $\mathbb{IR}^n$ iff, for each 'compute' step in the code list (25), the arguments to the basic interval operation e_i lie in the domain of e_i. If only the four BIAOs are used to define f, then every f defines the enclosure of a *rational function* of the inputs. $\mathcal{D}_f$ is the set of points $\mathbf{x}_0$ for which division by an interval containing zero does not occur while evaluating f at $[\mathbf{x}_0]$.

An interval expression is an *extension* of a real function if the interval expression produces the value of the real function when evaluated using degenerate interval arguments within the domain of the function. The following are interval extensions of the corresponding functions in (22a), (22b), and (22c) on page 14.

$$f_1([x_1], [x_2]) = [x_2] / ([x_1] + [x_2]) \tag{26a}$$

$$f_2([x_1], [x_2]) = 1 - 1/(1 + [x_2] / [x_1]) \tag{26b}$$

$$f_3([x_1], [x_2]) = 1/(1 + [x_1] / [x_2]) \tag{26c}$$

When different interval expressions are evaluated using non-degenerate interval arguments, they can produce different width intervals, although the resulting intervals must contain the set of all possible values of their respective underlying point expression. For (26a), (26b), and (26c), if interval inputs are not degenerate, only f_2 and f_3 always produce sharp bounds on their respective function's range over the domain subset defined by argument intervals. Multiple occurrences of arguments in f_1 can cause returned intervals to be unnecessarily wide. In the interval literature, this is known as 'the dependence problem', because interval arithmetic fails to recognize the two occurrences of the interval variable, $[X_2]$ in (26a), are the same variable and therefore dependent.

The Containment Set

The analysis of containment sets is connected to the presence or absence of a variable in an expression or sub-expression. Therefore, care is taken to ensure that the presence of a variable in the set of expression arguments means that the variable appears in the expression's definition. Recall item 6 on page 11 in the list of notation and term definitions.

Let the n_f arguments of the expression, $f\left(\left\{\mathbf{x}_f\right\}\right)$, be partitioned: $\mathbf{x}_f = \left(\mathbf{x}_h, \mathbf{x}_g\right)$. Further, let $h\left(\{\mathbf{x}_h\}\right)$ be an expression of the n_h variables, $\mathbf{x}_h = \left(\mathbf{x}_{h_u}, \mathbf{x}_c\right)$ and let $g\left(\left\{\left(y, \mathbf{x}_g\right)\right\}\right)$ be an expression of the $n_g + 1$ variables, $\left(y, \mathbf{x}_g\right) = \left(y, \mathbf{x}_{g_u}, \mathbf{x}_c\right)$. The only common arguments to both $\mathbf{x}_h$ and $\mathbf{x}_g$ are those in $\mathbf{x}_c$.

For a given $\mathbf{x}_h$ and $h\left(\{\mathbf{x}_h\}\right)$, consider all the possible compositions having the form,

$$f\left(\left\{\mathbf{x}_f\right\}\right) = g\left(\left\{\left(y, \mathbf{x}_g\right) \mid y \in h\left(\{\mathbf{x}_h\}\right)\right\}\right). \tag{27}$$

Depending on the form of the composition in (27), different members of $\mathbf{x}_h$ are in $\mathbf{x}_c$ and therefore in $\mathbf{x}_g$. Denote the containment set of h, evaluated at the point, $\mathbf{x}_{0h}$: $\mathrm{cset}\left(h, \{\mathbf{x}_{0h}\}\right)$. The interval evaluation, $h\left([\mathbf{x}_{0h}]\right)$, of the expression h must contain $\mathrm{cset}\left(h, \{\mathbf{x}_{0h}\}\right)$. The value of $\mathrm{cset}\left(h, \{\mathbf{x}_{0h}\}\right)$ can cause a containment failure to occur if there exists a composition having the form in (27) and

$$\mathrm{cset}\left(f, \left\{\mathbf{x}_{0f}\right\}\right) \not\subseteq \mathrm{cset}\left(g, \left\{\left(y, \mathbf{x}_{0g}\right) \mid y \in \mathrm{cset}\left(h, \{\mathbf{x}_{0h}\}\right)\right\}\right). \tag{28}$$

Relation (28) is a containment failure because the composition of containment sets on the right-hand side fails to contain all the elements in $\mathrm{cset}\left(f, \left\{\mathbf{x}_{0f}\right\}\right)$. This conception of a containment failure is motivated by the following considerations:

- Relation (28) is the basis for defining the *containment constraint* that containment sets must satisfy.
- Relation (28) is the essential event that containment sets must prevent. That is, when used as an argument of any *subsequent expression*, containment sets must not cause a containment failure.
- Unlike expression (18) on page 12 for the containment set over a set within the expression's domain, equation (28) begs the question of what containment sets

are. Consequently, equation (28) admits containment sets with arguments at singular points and indeterminate forms.

Definition 2 The *containment constraint* on the set of values, Y_0, of the expression h of n_h variables, evaluated at the point $\mathbf{x}_{0h}$ and denoted $h(\{\mathbf{x}_{0h}\})$, is that:

$$\operatorname{cset}\left(f, \left\{\mathbf{x}_{0f}\right\}\right) \subseteq \left\{ z \,\middle|\, \begin{array}{l} z \in g\left(\left\{\left(y_0, \mathbf{x}_{0g}\right)\right\}\right) \\ y_0 \in Y_0 \end{array} \right\}, \tag{29}$$

using any possible composition of the form,

$$\operatorname{cset}\left(f, \left\{\mathbf{x}_f\right\}\right) = g\left(\left\{\left(y, \mathbf{x}_g\right) \mid y \in h\left(\{\mathbf{x}_h\}\right)\right\}\right), \tag{30}$$

and any $\mathbf{x}_{0f} \in (I\!R^*)^{n_f}$, for which

$$\overline{f}\left(\left\{\mathbf{x}_{0f}\right\}\right) \neq \emptyset. \tag{31}$$

A trivial way to satisfy the containment constraint and therefore avoid containment failures is to let Y_0 in Definition 2, and therefore $\operatorname{cset}(h, \{\mathbf{x}_{0h}\})$, be the entire set of extended real numbers, $I\!R^*$. Because unnecessary members of $\operatorname{cset}(h, \{\mathbf{x}_{0h}\})$ are not wanted, the containment set of h at $(\mathbf{x}_{0h})$ must be the smallest set that satisfies the containment constraint in Definition 2.

The Central Problem of Extended Interval Analysis

For any extended interval $[\mathbf{X}_0] \subseteq (I\!R^*)^n$, the central problem of interval analysis in (4) on page 3 is solved with an enclosure, over the interval $[\mathbf{X}_0]$, for the containment set of f evaluated over all points in the interval $[\mathbf{X}_0]$:

$$f([\mathbf{X}_0]) \supseteq \operatorname{cset}(f, [\mathbf{X}_0]) \tag{32}$$

given

$$\operatorname{cset}(f, [\mathbf{X}_0]) = \left\{ z \,\middle|\, \begin{array}{l} z \in \operatorname{cset}(f, \{\mathbf{x}_0\}) \\ \mathbf{x}_0 \in [\mathbf{X}_0] \end{array} \right\}. \tag{33}$$

The next question to be answered is: Are there any additional restrictions that must be imposed to complete the containment set definition? The answer is yes. Containment sets must satisfy two conditions in addition to the containment constraint in Definition 2:

1. To satisfy the containment constraint, the containment set must contain any defined value of f at the point $\mathbf{x}_0$. Therefore, for any value of $\mathbf{x}_0$ within the domain of f, cset$(f, \{\mathbf{x}_0\})$ must contain all values of $f(\{\mathbf{x}_0\})$. If $\mathbf{x}_0$ is outside $\mathcal{D}_f$, either $f(\{\mathbf{x}_0\})$ may be empty, or if f is a relation, $f(\{\mathbf{x}_0\})$ may be a set of values.
2. Containment sets must have a kind of continuity. Suppose $(\mathbf{x}_{0j})$ is a sequence in $(I\!R)^n$ converging to $\mathbf{x}_0$. Choose an arbitrary $y_{0j} \in$ cset$(f, \{\mathbf{x}_{0j}\})$ for each j such that the y_{0j} converge to some y_0. Then y_0 must belong to cset$(f, \{\mathbf{x}_0\})$.

Condition 2 is not as obvious as condition 1, but is no less reasonable. In effect, condition 2 requires containment sets to use the topology of the system within which distance between points and convergence of sequences are defined. It would make no sense if containment sets used a different topology.

Interestingly, the identity of containment sets and closures follows directly from Condition 1 and 2, alone.

Lemma 1 Given the expression (function or relation), f, of n variables $\mathbf{x} = (x_1, \cdots, x_n)$ with domain, D_f, the point, $\mathbf{x}_0$ that may be outside D_f, and the following two constraints on the relation, $Y(\{\mathbf{x}_0\})$:

i. For all $\mathbf{x}_0 \in D_f$, then the set $Y(\{\mathbf{x}_0\})$ must satisfy: $Y(\{\mathbf{x}_0\}) \supseteq f(\{\mathbf{x}_0\})$.

ii. For any sequence $(\mathbf{x}_{0j})$ with $\mathbf{x}_{0j} \in D_f$ that converges to $\mathbf{x}_0$, and any sequence (y_{0j}), whose members satisfy: $y_{0j} \in Y(\{\mathbf{x}_{0j}\}) \supseteq f(\{\mathbf{x}_{0j}\})$ and converge to some value, y_0, then $y_0 \in Y(\{\mathbf{x}_0\})$.

Then $Y(\{\mathbf{x}_0\}) = \overline{f}(\{\mathbf{x}_0\})$, the closure of f, evaluated at the point $\mathbf{x}_0$.

Proof. The first step is to prove that $Y(\{\mathbf{x}_0\})$ must contain $\overline{f}(\mathbf{x}_0)$. Begin with any $y_0 \in \overline{f}(\mathbf{x}_0)$. By definition y_0 is the limit of a sequence (y_{0j}) where $y_{0j} \in f(\{\mathbf{x}_{0i}\})$ for some $\mathbf{x}_{0j}$ whose limit is $\mathbf{x}_0$. From applying condition *i* to the members of the sequence $(\mathbf{x}_{0i})$, it follows that $f(\{\mathbf{x}_{0j}\}) \subseteq Y(\{\mathbf{x}_{0j}\})$ so any $y_{0j} \in Y(\{\mathbf{x}_{0j}\})$. From condition *ii*, it follows that $y_0 \in Y(\{\mathbf{x}_{0j}\})$, completing the first step.

The second step is to prove that expression closures satisfy conditions *i* and *ii*. From Definition 1 of an expression closure on page 15, $\overline{f}(\{\mathbf{x}_0\})$ contains all defined values of f at $\mathbf{x}_0$, thereby satisfying condition i. Condition ii is also satisfied, because an expression's closure is the set of all possible accumulation points in the subsequences whose members are elements of the sequence of sets, $\left(f\left(\left\{\mathbf{x}_j\right\}\right)\right)$, for all subsequences of any sequence $\left(\mathbf{x}_j\right)$ with common accumulation point, $\mathbf{x}_0$. This completes the second step and the proof. ∎

Because the hypotheses of Lemma 1 are containment-set conditions, it follows at once that containment sets must be simply expression closures. While satisfying the containment constraint is the motivation for defining the containment set of an expression, Lemma 1 and the hypotheses therein are sufficient to require the containment set of an expression to be the expression's closure.

It remains to prove that expression closures satisfy the containment constraint. Because Lemma 1 requires an expression's containment set to be the expression's closure, if closures satisfy the containment constraint, they are the smallest sets that do so.

To prove expression closures satisfy the containment constraint, start with simple compositions having the form:

$$f(\{\mathbf{x}\}) = g(h(\{\mathbf{x}\})). \tag{34}$$

Lemma 2 establishes that closures of the compositions in (34) are subset-equal to the corresponding composition of their closures. In Lemma 3, Lemma 2 is generalized to compositions having more general form in (35):

$$f(\{\mathbf{x}\}) = g(\{(y, \mathbf{x}) \mid y \in h(\{\mathbf{x}\})\}) \tag{35}$$

from which any arbitrary expression can be built up. Note that because closures of expressions are defined at all points $\mathbf{x}_0 \in (I\!R^*)^n$ (see Definition 1, page 15), Lemma 2 holds, even if $\overline{f}(\{\mathbf{x}_0\}) = \emptyset$.

Lemma 2 Given the composite expression, f, of n variables, with $f(\{\mathbf{x}\}) = g(h(\{\mathbf{x}\}))$, then for any $\mathbf{x}_0 \in (I\!R^*)^n$, the closure of f at $\mathbf{x}_0$ satisfies the following subset inequality:

$$\overline{f}(\{\mathbf{x}_0\}) \subseteq \overline{g}\left(\left\{y_0 \mid y_0 \in \overline{h}(\{\mathbf{x}_0\})\right\}\right). \tag{36}$$

Proof. Take any $z_0 \in \overline{f}(\{\mathbf{x}_0\})$. So $z_0 = \lim_{j\to\infty} z_j$ where $z_j \in g(\{y_j\})$ for some $y_j \in h(\{\mathbf{x}_j\})$, and for some sequence $\left(\mathbf{x}_j\right)$ which converges to $\mathbf{x}_0$.

By compactness of $I\!R^*$, and by taking a subsequence, assume[6] that the y_j converge to some y_0. Then by the definition of the closure of a relation, $z_0 \in \overline{g}(\{y_0\})$. Similarly $y_0 \in \overline{h}(\{\mathbf{x}_0\})$. So $z_0 \in \overline{g}(\{y_0 \mid y_0 \in \overline{h}(\{\mathbf{x}_0\})\})$, proving the result. ∎

The Analysis of Dependence

Extending Lemma 2 for the simple compositions in (36) to the general form in (35) requires an analysis of *dependence* between multiple occurrences of expression arguments. The term dependence in the interval literature is used to describe an expression in which at least one argument is used more than once. Because interval arithmetic does not recognize dependence, every expression is evaluated as if each occurrence of a variable is independent of every other occurrence of the same variable. For example, instead of computing a sharp enclosure of the expression:

$$\frac{x}{x+y} \tag{37}$$

at the point (x_0, y_0), interval arithmetic computes a sharp enclosure of the expression:

$$\frac{z}{x+y} \tag{38}$$

at the point $(x_0, y_0, z_0) = (x_0, y_0, x_0)$.

In the following three sub-sections, the range of possible dependencies in a single composition is examined.

The General Case.

Let the n_f arguments of the expression, $f(\{\mathbf{x}_f\})$, be partitioned: $\mathbf{x}_f = (\mathbf{x}_{h_u}, \mathbf{x}_{g_u}, \mathbf{x}_c)$. Further, let $h(\{\mathbf{x}_h\})$ be an expression of the n_h variables, $\mathbf{x}_h = (\mathbf{x}_{h_u}, \mathbf{x}_c)$ and let $g(\{(y, \mathbf{x}_g)\})$ be an expression of the $n_g + 1$ variables, $(y, \mathbf{x}_g) = (y, \mathbf{x}_{g_u}, \mathbf{x}_c)$. The arguments in $\mathbf{x}_c$ are the only common arguments to both $\mathbf{x}_h$ and $\mathbf{x}_g$. The arguments in $\mathbf{x}_{h_u}$ and $\mathbf{x}_{g_u}$ appear only in h and g, respectively. Whether there are common arguments within $\mathbf{x}_h$ and $\mathbf{x}_g$ is unspecified for the time being.

[6] This is the customary compressed way to express the following argument. Recall that a subsequence means a sequence y_{j_k} where $j_1, j_2, j_3, \ldots$ is some strictly increasing sequence of indices. Take the *corresponding* subsequences of the z_js and $\mathbf{x}_j$s, namely z_{j_k} and $\mathbf{x}_{j_k}$. These still converge to z_0 and $\mathbf{x}_0$ respectively. So, renaming these subsequences with the original names (i.e. naming $\mathbf{x}_{j_k}$ as $\mathbf{x}_j$, etc.) results in sequences with all the original properties and the extra property that the y_js converge to y_0.

The Single-Use Case.

One extreme is the case when there are no arguments in $\mathbf{x}_c$. Then the n_f arguments of the expression, $f\left(\left\{\mathbf{x}_f\right\}\right)$, can be partitioned such that $\mathbf{x}_f = \left(\mathbf{x}_h, \mathbf{x}_g\right)$, with $h\left(\{\mathbf{x}_h\}\right)$ an expression of the n_h variables $\mathbf{x}_h$, and $g\left(\left\{\left(y, \mathbf{x}_g\right)\right\}\right)$ an expression of the $n_g + 1$ variables, $\left(y, \mathbf{x}_g\right)$. There are no common $\mathbf{x}_c$ arguments in $\mathbf{x}_h$ and $\mathbf{x}_g$.

The Total Dependence Case.

The other extreme is when all arguments are in $\mathbf{x}_c$ and there are no unique arguments, $\mathbf{x}_{h_u}$ or $\mathbf{x}_{g_u}$, associated with h or g. In this case:

$$\mathbf{x}_f = \mathbf{x}_h = \mathbf{x}_c = \mathbf{x}_g, \tag{39}$$

and $f\left(\left\{\mathbf{x}_{f0}\right\}\right) = g\left(\left\{\left(h\left(\left\{\mathbf{x}_{f0}\right\}\right), \mathbf{x}_{f0}\right)\right\}\right)$, so no partitioning of the arguments of f is needed.

Regardless of the degree of dependence, Lemma 2 extends to the general composition in (35):

Lemma 3 Let the n_f arguments of the expression, $f\left(\left\{\mathbf{x}_f\right\}\right)$, be partitioned: $\mathbf{x}_f = \left(\mathbf{x}_{h_u}, \mathbf{x}_{g_u}, \mathbf{x}_c\right)$. Further, let $h\left(\{\mathbf{x}_h\}\right)$ be an expression of the n_h variables, $\mathbf{x}_h = \left(\mathbf{x}_{h_u}, \mathbf{x}_c\right)$ and let $g\left(\left\{\left(y, \mathbf{x}_g\right)\right\}\right)$ be an expression of the $n_g + 1$ variables, $\left(y, \mathbf{x}_g\right) = \left(y, \mathbf{x}_{g_u}, \mathbf{x}_c\right)$. The members of $\mathbf{x}_c$ are the only common arguments to both $\mathbf{x}_h$ and $\mathbf{x}_g$. The arguments in $\mathbf{x}_{h_u}$ and $\mathbf{x}_{g_u}$ appear only in h and g, respectively.

Consider the composition having the form,

$$f\left(\mathbf{x}_f\right) = g\left(\left\{\left(y, \mathbf{x}_g\right) \mid y \in h\left(\{\mathbf{x}_h\}\right)\right\}\right). \tag{40}$$

Then regardless of the number of common arguments in $\mathbf{x}_c$,

$$\overline{f}\left(\left\{\mathbf{x}_{f0}\right\}\right) \subseteq \overline{g}\left(\left\{\left(y_0, \mathbf{x}_{g0}\right) \mid y_0 \in \overline{h}\left(\{\mathbf{x}_{h0}\}\right)\right\}\right). \tag{41}$$

The proof parallels that of Lemma 2, using the single-use and total-dependence cases defined above.

Proof. Take any $z_0 \in \overline{f}(\{\mathbf{x}_{f0}\}) = \overline{f}(\{\left(\mathbf{x}_{h_u0}, \mathbf{x}_{g_u0}, \mathbf{x}_{c0}\right)\})$. From Definition 1, $z_0 = \lim_{j\to\infty} z_j$ where $z_j \in g(\left\{\left(y_j, \mathbf{x}_{gj}\right)\right\}) = g(\left\{\left(y_j, \mathbf{x}_{g_uj}, \mathbf{x}_{cj}\right)\right\})$ for some $y_j \in h(\{\mathbf{x}_{hj}\}) = h(\{\left(\mathbf{x}_{h_uj}, \mathbf{x}_{cj}\right)\})$, and for some sequence, $\left(\mathbf{x}_{fj}\right) = \left(\mathbf{x}_{h_uj}, \mathbf{x}_{g_uj}, \mathbf{x}_{cj}\right)$, which converge to $\mathbf{x}_{f0}$.

In the single-use case, $\mathbf{x}_c$ contains no members. Except for the fact that g has arguments, $\mathbf{x}_g$, in addition to y, this is precisely the same situation as that in Lemma 2. The expression g depends on the members of $\mathbf{x}_h$ only through h. By compactness of $I\!R^*$, and by taking a subsequence, assume that the y_j converge to some y_0. Then by the definition of the closure of a relation, $z_0 \in \overline{g}(\{(y_0, \mathbf{x}_{g0})\})$. Similarly $y_0 \in \overline{h}(\{\mathbf{x}_{h0}\})$. So $z_0 \in \overline{g}(\{(y_0, \mathbf{x}_{g0}) \mid y_0 \in \overline{h}(\{\mathbf{x}_{h0}\})\})$, proving the result in the single-use case.

In the total-dependence case, equation (39) is true. When computing the closure of f, members of $\mathbf{x}_c$ that are common to both g and h are taken into account. In contrast, when computing the composition of closures, the fact that the members of $\mathbf{x}_c$ are common to g and h is ignored. The composition of closures can take on more values than the closure of f because the members of $\mathbf{x}_c$ in g and h are free to vary independently. This case is the same as the single-use case, with equal *values* of the variables in $\mathbf{x}_g$ and $\mathbf{x}_h$. That is, $\mathbf{x}_{g0} = \mathbf{x}_{h0}$, but with the variables themselves unequal – recall item 1 on page 8. The remainder of the proof in this case is exactly the same as in the single-use case.

Since the two extreme cases bound all possible degrees of dependence, the required result is proved. ∎

Example 2 From Lemma 3, the following expression closures of the functions in (22a), (22b), and (22c) are subset equal to the corresponding compositions of BAO closures:

$$\overline{f}_1(\{x_1\}, \{x_2\}) \subseteq \{x_2\} \overline{/} (\{x_1\} \overline{+} \{x_2\}) \tag{42a}$$

$$\overline{f}_2(\{x_1\}, \{x_2\}) \subseteq 1 \overline{-} 1 \overline{/} (1 \overline{+} \{x_2\} \overline{/} \{x_1\}) \tag{42b}$$

$$\overline{f}_3(\{x_1\}, \{x_2\}) \subseteq 1 \overline{/} (1 \overline{+} \{x_1\} \overline{/} \{x_2\}) \tag{42c}$$

The values in Table 6 illustrate how function values, expression closures and compositions of BAO closures are related. The first row contains column headings. The first column contains argument values. The second column contains defined function values for f_1, f_2, and f_3 if arguments are contained in the function's natural domain. Question marks denote undefined values. "na" means not applicable. Column 3 contains the value of the expression's closure. Note that expression closures are always defined and that all three expressions have the same closure values. From the identity of containment sets and expression closures, these values are the containment sets of the expressions that must be enclosed by their interval evaluation. Column 4 contains the values of the right-hand sides of (42a), (42b), and

(x_1, x_2)	(f_1, f_2, f_3)	$(\overline{f}_1, \overline{f}_2, \overline{f}_3)$	expression clsosure compositions
$(1, 1)$	$\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$	$\left\{\frac{1}{2}\right\}$	$\left\{\frac{1}{2}\right\}$
$(1, 0)$	$(0, 0, ?)$	$\{0\}$	$\{0\}$
$(0, 1)$	$(1, ?, 1)$	$\{1\}$	$\{1\}$
$(-1, 1)$	$(?, ?, ?)$	$\{-\infty, +\infty\}$	$\{-\infty, +\infty\}$
$(0, 0)$	$(?, ?, ?)$	$I\!R^*$	$I\!R^*$
$(+\infty, +\infty)$	$(?, ?, ?)$	$[0, 1]$	$([0, +\infty], [0, 1], [0, 1])$
$(\{1, 2\}, \{1, 2\})$	na	$\left\{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}\right\}$	$\left(\left\{\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 1\right\}, \left\{\frac{1}{3}, \frac{2}{3}\right\}, \left\{\frac{1}{3}, \frac{2}{3}\right\}\right)$
$([1, 2], [1, 2])$	na	$\left[\frac{1}{3}, \frac{2}{3}\right]$	$\left(\left[\frac{1}{4}, 1\right], \left[\frac{1}{3}, \frac{2}{3}\right], \left[\frac{1}{3}, \frac{2}{3}\right]\right)$

TABLE 6 Sample values of expressions (42a), (42b), and (42c).

(42c). As Lemma 3 predicts, subset equality holds between expression closures and compositions of BAO closures. As expected, compositions of BAO closures for f_1 may include unneeded values because the composition of BAO closures does not recognize dependence between multiple occurrences of x_2.

Containment-Set Closure Identity

The containment-set closure identity follows at once from Lemmas 1 and 3:

Theorem 1 Given any expression $h(\{\mathbf{x}\})$ of n variables and the point, $\mathbf{x}_0$, then the containment set,

$$\text{cset}(h, \{\mathbf{x}_0\}) = \overline{h}(\{\mathbf{x}_0\}) \tag{43}$$

is the smallest set that satisfies the containment constraint and conditions i and ii in Lemma 3.

Proof. Expression closures are uniquely determined by Lemma 1. Lemma 3 guarantees that expression closures satisfy the containment constraint in Definition 2. ∎

Theorem 1 establishes that the containment set and closure of any expression are identical. Therefore, containment sets inherit all the properties of expression closures.

Basic Arithmetic Operation Containment Sets

Having established that containment sets are closures, the containment sets of the BAOs and other intrinsic functions are established using Theorem 1, the containment-set closure identity, and Definition 1 of the closure of an expression.

Addition

Let $f_+(x, y) = x + y$. Then from the containment-set closure identity and the definition of an expression closure,

$$\text{cset}\,(f_+, \{(x_0, y_0)\}) = \overline{f}_+(\{(x_0, y_0)\}) \tag{44a}$$

$$= \left\{ z \;\middle|\; \begin{array}{l} \lim_{j\to\infty} x_j = x_0 \\ \lim_{j\to\infty} y_j = y_0 \\ \lim_{j\to\infty} \left(x_j + y_j\right) = z \end{array} \right\} \tag{44b}$$

Points that are in $\overline{f}_+$, but not in f_+, include those values for which at least one of x_0 or y_0, must be $-\infty$ or $+\infty$. Without loss of generality (because addition commutes), let $x_0 \in \{-\infty, +\infty\}$.

- If $x_0 = \infty$ and $y_0 \in \mathbb{R}$, then $x_j + y_j \to \infty$.
- If $x_0 = \infty$ and $y_0 = \infty$, then $x_j + y_j \to \infty$.
- If $x_0 = \infty$ and $y_0 = -\infty$, then $x_j + y_j$ can approach any finite or infinite value. For example:
  - To get any finite z, let $y_j = z - x_j$.
  - To get $z = -\infty$, let $y_j = -x_j^2 - x_j$.
  - To get $z = +\infty$, let $x_j = y_j^2 - y_j$.

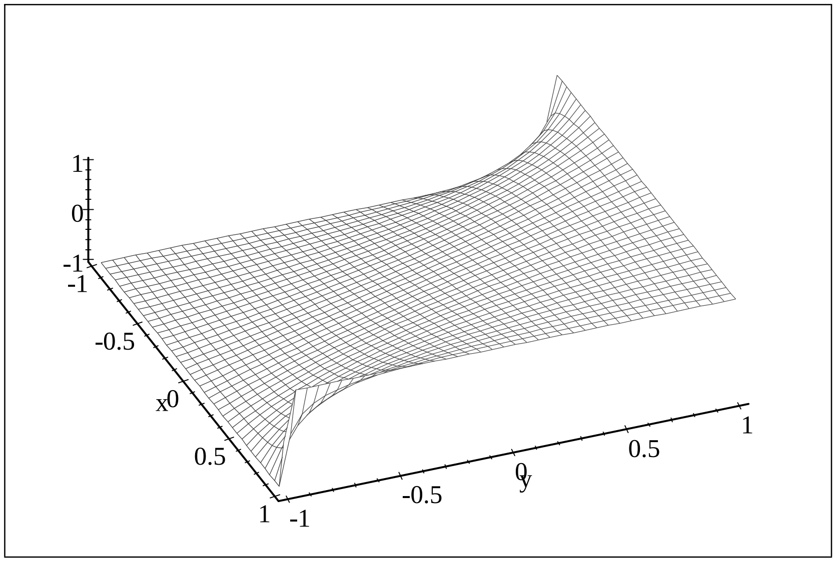

FIGURE 1 $u_0 = \overline{\tanh}(\{x_0\})\,,\ v_0 = \overline{\tanh}(\{y_0\})\,,$
$w_0 = \mathrm{cset}(\tanh(x+y)\,, \{x_0, y_0\})\,.$

The case of $x_0 = -\infty$ is similar. Therefore, the following closure of addition and subtraction as shown in Tables 2 and 3 is justified for $x_0 \in \{-\infty, +\infty\}$:

$$\mathrm{cset}(x+y, \{(-\infty, y_0)\}) = \{-\infty\}\,,\ \text{for } y_0 < +\infty,$$
$$\mathrm{cset}(x+y, \{(+\infty, y_0)\}) = \{+\infty\}\,,\ \text{for } y_0 > -\infty,$$
$$\mathrm{cset}(x+y, \{(-\infty, y_0)\}) = I\!R^*,\ \text{for } y_0 = +\infty.$$

The graph in Figure 1 depicts the closure of addition. The u-, v-, and w-axes are the mappings of x_0, y_0, and $\mathrm{cset}(x+y, \{(x_0, y_0)\})$ onto the interval $[-1, 1]$ using the hyperbolic tangent map. To properly illustrate that $\mathrm{cset}(x+y, \{(-\infty, +\infty)\}) = I\!R^*$, the graph should actually contain vertical lines from $(u, v, w) = (1, -1, -1)$ to $(1, -1, 1)$ and from $(-1, 1, -1)$ to $(-1, 1, 1)$.

Multiplication

Let $f_\times(x, y) = x \times y$. Then following the same justification as in the case of addition,

$$\text{cset}(f_\times, \{(x_0, y_0)\}) = \overline{f}_\times(\{(x_0, y_0)\}) \tag{45a}$$

$$= \left\{ z \;\middle|\; \begin{array}{l} \lim_{j\to\infty} x_j = x_0 \\ \lim_{j\to\infty} y_j = y_0 \\ \lim_{j\to\infty} \left(x_j \times y_j\right) = z \end{array} \right\} \tag{45b}$$

Points that are in $\overline{f}_\times$, but not in $f_\times$, include those values for which at least one of x_0 or y_0 must be $-\infty$ or $+\infty$. Without loss of generality (because multiplication commutes), let $x_0 \in \{-\infty, +\infty\}$.

- If $x_0 = \infty$, $y_0 \in \mathbb{R}$ and $y \neq 0$, then:

$$x_j \times y_j \to \begin{cases} -\infty, & \text{if } y < 0 \\ +\infty, & \text{if } y > 0 \end{cases}.$$

- If $x_0 = \infty$ and $y_0 \in \{-\infty, +\infty\}$, then the results in the previous case hold.
- If $x_0 = \infty$ and $y_0 = 0$, then $x_j \times y_j$ can approach any finite or infinite value. For example:
  - To get any finite z, let $y_j = \frac{z}{x_j}$.
  - To get $z = -\infty$, let $y_j = \frac{1}{-\sqrt{x_j}}$.
  - To get $z = +\infty$, let $y_j = \frac{1}{+\sqrt{x_j}}$.

The case of $x_0 = -\infty$ is similar. Therefore, the following closure of multiplication as shown in Table 4 is justified for $x_0 \in \{-\infty, +\infty\}$:

$$\text{cset}(x \times y, \{(x_0, y_0)\}) = \{\text{signum}(x_0) \times \text{signum}(y_0) \times \infty\}, \text{ for } y \neq 0 \text{ and}$$

$$\text{cset}(x \times y, \{(x_0, y_0)\}) = \mathbb{R}^*, \text{ for } y_0 = 0.$$

Similar to Figure 1, Figure 2 depicts the closure of multiplication using the hyperbolic tangent mapping of x_0, y_0, and $\text{cset}(x \times y, \{x_0, y_0\})$ onto the u-, v-, and w-axes, respectively. To properly illustrate that $\text{cset}(x \times y, \{(0, \pm\infty)\}) = \mathbb{R}^*$, the graph should actually contain vertical lines from $w = -1$ to $w = +1$ at $(x_0, y_0) = (0, -1)$, $(1, 0)$, $(0, 1)$, and $(-1, 0)$.

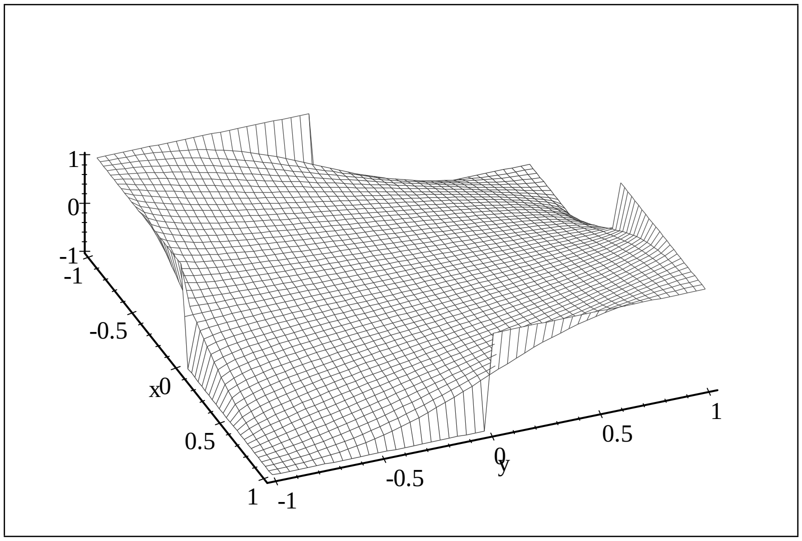

FIGURE 2 $u_0 = \overline{\tanh}(\{x_0\})$, $v_0 = \overline{\tanh}(\{y_0\})$, $w_0 = \text{cset}(\tanh(x \times y), \{x_0, y_0\})$.

Division

Let $f_{\div} = (x \div y)$ then

$$\text{cset}(f_{\div}, \{(x_0, y_0)\}) = \overline{f}_{\div}(\{(x_0, y_0)\}) \tag{46a}$$

$$= \left\{ z \;\middle|\; \begin{array}{l} \lim_{j\to\infty} x_j = x_0 \\ \lim_{j\to\infty} y_j = y_0 \\ \lim_{j\to\infty} \left(x_j \div y_j\right) = z \end{array} \right\}. \tag{46b}$$

Points that are in $\overline{f}_{\div}$, but not in $f_{\div}$, include those values for which at least one of x_0 or y_0 must be $-\infty$, $+\infty$, or $y_0 = 0$.

- If $x_0 = \infty$, $y_0 \in I\!R$, and $y \neq 0$, then

$$\lim_{j\to\infty} \left(x_j \div y_j\right) = \begin{cases} -\infty, & \text{if } y < 0 \\ +\infty, & \text{if } y > 0 \end{cases}.$$

Similarly, if $x_0 \in \{-\infty, +\infty\}$, $x_j \div y_j = \text{signum}(x_0) \times \text{signum}(y_0) \times (+\infty)$.

- If $x_0 = +\infty$ and $y_0 = +\infty$, then $x_j \div y_j$ can approach any non-negative finite or infinite value. For example:

  - To get $z_0 = 0$, let $y_j = x_j^2$.
  - To get any finite $0 < z_0 < \infty$, let $y_j = \frac{x_j}{z}$.
  - To get $z_0 = +\infty$, let $x_j = y_j^2$.
  - To prove z_0 cannot be negative, it is sufficient to prove that $\lim_{j\to\infty}\left(\frac{x_j}{y_j}\right) \neq z_0$ if $z_0 < 0$.

    Proof. For sufficiently large j, say $j > j_0$, both x_j and y_j are positive. Thus, for $j > j_0$, $\frac{x_j}{y_j} \geq 0$ and $\left|\frac{x_j}{y_j} - z_0\right| \geq |z_0| > 0$. Therefore $\frac{x_j}{y_j}$ cannot approach $z_0 < 0$. ∎

  Similarly, if $x_0 \in \{-\infty, +\infty\}$ and $y_0 \in \{-\infty, +\infty\}$, then $\lim_{j\to\infty}(x_j \div y_j) = \text{signum}(x_0) \times \text{signum}(y_0) \times [0, \infty]$.

- If $x_0 \in I\!R$ and $y_0 \in \{-\infty, +\infty\}$, then $\lim_{j\to\infty}(x_j \div y_j) = 0$. For example:

  - To get $z_0 = 0$, let $y_j \in \{\pm j x_j\}$.

- If $x_0 \in I\!R$, $x_0 > 0$, and $y_0 = 0$, then $\lim_{j\to\infty}(x_j \div y_j) \to \{\pm\infty\}$.

  - To get $z_0 = -\infty$, let $y_j = \frac{-x_j}{j}$.
  - To get $z_0 = +\infty$, let $y_j = \frac{x_j}{j}$.
  - To prove z_0 cannot be finite, it is sufficient to prove that $\frac{1}{z_0} = \frac{|y_j|}{x_j} \nrightarrow z_0$ if $z_0 \neq 0$.

    Proof. For sufficiently large j, say $j > j_0$, $x_j > 0$, $|y_j| < x_j$ and $\left|\frac{|y_j|}{x_j} - |z_0|\right| \geq |z_0| > 0$. Therefore $\frac{|y_j|}{x_j}$ cannot approach z_0 if $|z_0| > 0$. ∎

  Similarly, if $x_0 \in I\!R$, $x_0 < 0$, and $y_0 = 0$, then $\lim_{j\to\infty}(x_j \div y_j) = \{\pm\infty\}$.

- If $x_0 = +\infty$ and $y_0 = 0$, then $\lim_{j\to\infty}\left(x_j \div y_j\right) = \{\pm\infty\}$.

  - To get $z_0 = -\infty$, let $y_j = \frac{-1}{x_j}$.

  - To get $z_0 = +\infty$, let $y_j = \frac{1}{x_j}$.

  Similarly, if $x_0 = -\infty$, and $y_0 = 0$, then $\lim_{j\to\infty}\left(x_j \div y_j\right) = \{\pm\infty\}$

- If $x_0 = 0$ and $y_0 = 0$, then $x_j \div y_j$ can approach any finite or infinite value. For example:

  - To get $z_0 = -\infty$, let $y_j = -\operatorname{signum}\left(x_j\right) \times x_j^2$.

  - To get $z_0 = +\infty$, let $y_j = \operatorname{signum}\left(x_j\right) \times x_j^2$.

  - To get any finite $z_0 \neq 0$, let $y_j = \frac{x_j}{z}$.

  - To get $z_0 = 0$, let $\left|x_j\right| = y_j^2$.

Therefore, the following closure of division as shown in Table 5 is justified:

$$\operatorname{cset}(x \div y, \{(x_0, y_0)\}) = \{0\}, \text{ for } x_0 \in I\!R \text{ and } y_0 \in \{-\infty, +\infty\},$$

$$\operatorname{cset}(x \div y, \{(x_0, y_0)\}) = \begin{array}{l} \operatorname{signum}(y_0) \times \operatorname{signum}(y_0) \times [0, \infty], \\ \text{for } x_0 \text{ and } y_0 \in \{-\infty, +\infty\}, \end{array}$$

$$\operatorname{cset}(x \div y, \{(x_0, 0)\}) = \{\pm\infty\}, \text{ for } x_0 \in \overline{I\!R}_{\div} \text{ and } x_0 \neq 0, \text{ and}$$

$$\operatorname{cset}(x \div y, \{(0, 0)\}) = I\!R^*.$$

The graph in Figure 2 depicts the closure of division using the same hyperbolic tangent mapping employed for addition and multiplication. To properly illustrate that $\operatorname{cset}(x \div y, \{(+\infty, +\infty)\}) = \operatorname{cset}(x \div y, \{(-\infty, -\infty)\}) = [0, \infty]$, and $\operatorname{cset}(x \div y, \{(-\infty,$ $\operatorname{cset}(x \div y, \{(+\infty, -\infty)\}) = [-\infty, 0]$ the graph should actually contain vertical lines from $w_0 = -1$ to $w_0 = 0$ at $(u_0, v_0) = (-1, -1)$, and $(1, 1)$, and from $w_0 = 0$ to $w_0 = 1$ at $(u_0, v_0) = (-1, 1)$, and $(1, -1)$. Finally, to properly illustrate that $\operatorname{cset}(x \div y, \{(0, 0)\}) = [-\infty, +\infty]$, there should be a vertical line from $(u_0, v_0, w_0) = (0, 0, -1)$ to $(0, 0, 1)$.

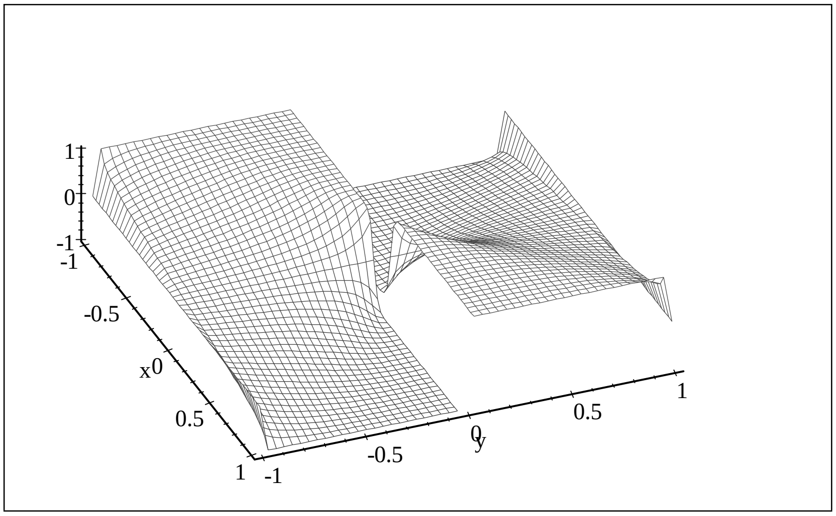

FIGURE 3 $u_0 = \overline{\tanh}(\{x_0\})$, $v_0 = \overline{\tanh}(\{y_0\})$, $w_0 = \text{cset}(\tanh(x \div y), \{x_0, y_0\})$.

Irrational Expressions

Similar treatments is used to define the closure of irrational expressions.

ln

In this case,

$$\text{cset}(\ln, \{x_0\}) = \overline{\ln}(\{x_0\}) \tag{47a}$$

$$= \left\{ y \;\middle|\; \begin{array}{l} \lim_{j\to\infty} x_j = x_0 \\ \lim_{j\to\infty} \ln x_j = z \end{array} \right\}. \tag{47b}$$

Points that are in $\overline{\ln}$, but not in ln, include those values for which x_0 is 0 or $+\infty$.

- If $x_0 = 0$, then $\lim_{j\to\infty} \ln x_j = -\infty$.

- If $x_0 = \infty$, then $\lim_{j\to\infty} \ln x_j = \infty$.

Therefore, the following closure of the natural logarithm is justified for $x_0 \in \{0, +\infty\}$:

$$\text{cset}\,(\ln x, \{0\}) = \{-\infty\}\,, \text{ and}$$
$$\text{cset}\,(\ln x, \{+\infty\}) = \{+\infty\}\,.$$

exp

In a fashion similar to the natural logarithm, the following closure of the exponential function is justified for $x_0 \in \{-\infty, +\infty\}$:

$$\text{cset}\,(\exp x, \{-\infty\}) = 0, \text{ and}$$
$$\text{cset}\,(\exp x, \{+\infty\}) = +\infty.$$

$\exp(y \ln x)$

Let $f_{\exp(y\ln x)} = \exp(y \ln x)$. Points that are in $\overline{f}_{\exp(y\ln x)}$, but not in $f_{\exp(y\ln x)}$, include those values for which $y_0 \in \{-\infty, 0, +\infty\}$ and $x_0 \in \{0, 1, +\infty\}$, excluding $(y_0, x_0) = (0, 1)$.

- If $y_0 = \infty$, $x_0 \in I\!R$ and $x \neq 1$, then

$$\exp\left(y_j \ln x_j\right) \to \begin{cases} 0, \text{ if } x < 1 \\ +\infty, \text{ if } x > 1 \end{cases}.$$

- If $y_0 = \infty$ and $x_0 \in \{0, +\infty\}$, then the results in the previous case hold.
- If $y_0 = \infty$ and $x_0 = 1$, then $\exp\left(y_j \ln x_j\right)$ can approach any non-negative finite or infinite value. For example:
  - To get any finite z_0, let $x_j = \exp\left(\frac{\ln z_0}{y_j}\right)$.
  - To get $z_0 = 0$, let $x_j = \exp\left(\frac{1}{-\sqrt{y_j}}\right)$.
  - To get $z_0 = +\infty$, let $x_j = \exp\left(\frac{1}{+\sqrt{y_j}}\right)$.

The cases of $y_0 \in \{-\infty, 0\}$ are similar. Therefore, the following closure of $\exp(y \ln x)$ is justified for $y_0 \in \{-\infty, 0, +\infty\}$ and $x_0 \in \{0, 1, +\infty\}$, excluding $(y_0, x_0) = (0, 1)$. Including $y_0 = 0$ is needed because $\overline{\ln}(\{x_0\})$ can be infinite when $x_0 \in \{0, +\infty\}$.

- For $y_0 \in \{-\infty, +\infty\}$ and $x_0 \in \{0, +\infty\}$,

$$\text{cset}(\exp(y \ln x), \{(x_0, y_0)\}) = \{\exp(\text{signum}(y) \times \text{signum}(x-1) \times \infty)\}. \tag{48}$$

- For $y_0 \in \{-\infty, +\infty\}$ and $x_0 = 1$, or $y_0 = 0$ and $x_0 \in \{0, +\infty\}$,

$$\text{cset}(\exp(y \ln x), \{(x_0, y_0)\}) = [0, +\infty]. \tag{49}$$

Variable and Value Equality

Theorem 1 establishes the identity of containment sets and closures. Therefore, the distinction between the equality of variables as contrasted with equality only of their values applies both to containment sets and closures. For example, $\text{cset}(x - x, \{x_0\})$ (or equivalently, $\text{cset}(x - y, \{(x_0, y_0) \,|x = y\})$), and $\text{cset}(x - y, \{(x_0, y_0) \,|x_0 = y_0\})$ are different. The following examples are illustrative.

$$\text{cset}(x - x, \{x_0\}) = \text{cset}(x - y, \{(x_0, x_0)\} \mid x = y) = \{0\} \tag{50a}$$

$$\text{cset}\left(x \times \left(\frac{1}{x}\right), \{x_0\}\right) = \text{cset}\left(x \times \left(\frac{1}{y}\right), \{(x_0, x_0)\} \mid x = y\right) = \{1\} \tag{50b}$$

$$\text{cset}\left(\frac{x}{x}, \{x_0\}\right) = \text{cset}\left(\frac{x}{y}, \{(x_0, x_0)\} \mid x = y\right) = \{1\}. \tag{50c}$$

Alternatively,

$$\text{cset}(x - y, \{(x_0, y_0) \,|x_0 = y_0\}) = \begin{cases} \{0\} & \text{for all } x_0 \in I\!R \\ I\!R^* & \text{if } x_0 \in \{-\infty, +\infty\} \end{cases}, \tag{51a}$$

$$\text{cset}\left(x \times \left(\frac{1}{y}\right), \{(x_0, x_0) \,|x_0 = y_0\}\right) = \begin{cases} \{1\} & \text{for all } x_0 \in I\!R - 0 \\ I\!R^* & \text{if } x_0 \in \{-\infty, 0, +\infty\} \end{cases}, \tag{51b}$$

and

$$\operatorname{cset}\left(\frac{x}{y}, (x_0, x_0) \,|\, x_0 = y_0\right) = \begin{cases} \{1\} & \text{for all } x_0 \in I\!R - 0 \\ I\!R^* & \text{if } x_0 = 0 \\ [0, +\infty] & \text{if } x_0 \in \{-\infty, +\infty\} \end{cases} . \tag{51c}$$

Containment-Set-Equivalent Expressions

Two expressions are *containment-set equivalent* if they have identical containment sets for all possible values of their arguments. The interval evaluation of containment-set-equivalent expressions produces an enclosure of their common containment set. Therefore, containment-set-equivalent expression exchange cannot cause a containment failure. This result can be used to choose the "best" containment-set-equivalent expression for a particular purpose.

Without loss of containment, expression h can replace expression f in any expression, if for all $\{\mathbf{x}_0\} \in (I\!R^*)^n$, $\operatorname{cset}(f, \{\mathbf{x}_0\}) \subseteq \operatorname{cset}(h, \{\mathbf{x}_0\})$.

Example 3 The functions, f_1, f_2, and f_3 on page 17 are containment-set-equivalent expressions. Therefore, the interval expression

$$f_2([X_1], [X_2]) \cap f_3([X_1], [X_2]) \tag{52}$$

is a sharp enclosure of the common containment set of the functions f_1, f_2, and f_3. In (52) f_1 is not needed, as the width of f_1 exceeds that of the intersection of f_2 and f_3.

Conclusion

Traditional interval analysis is defined for single-valued operations and functions with operands and arguments in their natural domains. Because intervals are sets, interval systems can be extended to:

1. permit interval argument endpoints to be any values in $I\!R^*$, whether partly or totally outside an expression's natural domain; and,

2. permit interval expressions to be enclosures either of functions or of relations.

The key new concept needed to make the required extensions is the *containment set* of possible results that an enclosure must contain, including argument values for which point expressions are not defined. The containment-set closure identity provides an operational definition of the containment set of any expression, whether a function or relation.

The practical consequences of these results are:

1. Interval arithmetic can be used to bound the range of relations as well as functions.
2. Closed interval systems can be implemented on a computer so that no undefined events, or IEEE exceptions, are logically possible.
3. Containment-set equivalence defines the set of expressions within which substitutions can be made without loss of containment.

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