Let C be a monoid. For $c, d \in C$, c is a prefix of d if there exists $e \in C$ such that $ce = d$. If C is group embeddable and $C^\times = \{c \in C : \exists d \in C \; cd = 1\}$ is trivial then C is a *sector*.

Let C, D be sectors. A map $e: C \to D$ is a *calc* if $e(1) = 1$ and $e(c)$ is a prefix of $e(cc')$ for all $c, c' \in C$. The *shift* of a calc e by an element $c \in C$ is the map $e^c = f: C \to D$ such that $e(c)f(c') = e(cc')$ for all $c' \in C$. It is a fact that the shift of a calc is a calc, $e^1 = e$, and $e^{cc'} = (e^c)^{c'}$, so C acts on $\{e: C \to C : e \text{ is a calc}\}$ on the right by shift.

Chain rule. Let C be a sector and $e, f: C \to C$ calcs. Let $g = (e \circ f)^c$ and $h = e^{f(c)} \circ f^c$. We have

$$e(f(c))g(c') = e(f(cc')) = e(f(c)f^c(c')) = e(f(c))e^{f(c)}(f^c(c')),$$

so canceling by $e(f(c))$ yields $g(c') = e^{f(c)}(f^c(c')) = h(c')$ for all $c' \in C$.

Galois group. Let E be a set of calcs with domain and codomain C. The Galois group $\mathrm{Gal}(E)$ is the group $\{\varphi : \varphi \circ e = e \circ \varphi \; \forall e \in E\}$ of monoid automorphisms of C with multiplication composition of maps.

Exercise. Let $C = \langle c_1, \ldots, c_n \rangle$. It is a fact that every automorphism φ of C permutes the generators of C. For every $c \in C$, there is a unique way to write c as a product of generators, and the number of generators, counted with occurrences, is the *length* of c. For $c \in C$ and n a non-negative integer, $\mathrm{trunc}(c, n)$ is the prefix of c of length n, if there is one, and c otherwise. An element $c \in C$ *follows a σ pattern*, for $\sigma \in S_n$, if there is some non-negative integer k such that $c = c_i \sigma(c_i) \sigma^2(c_i) \cdots \sigma^k(c_i)$ for some generator c_i. For $c \in C$ and $\sigma \in S_n$, let $\mathrm{iter}_\sigma(c)$ be the greatest positive integer k such that $c_i \sigma(c_i) \sigma^2(c_i) \cdots \sigma^k(c_i)$ is a prefix of c for some generator c_i. So, $\mathrm{iter}_\sigma(c)$ is the length of the longest prefix of c that follows a σ-pattern.

Let $\sigma \in S_n$ be a permutation and e_σ the calc with domain and codomain C defined by $e_\sigma(c) = \mathrm{trunc}(c, \mathrm{iter}_\sigma(c))$. So e_σ just emits generators it reads from its input left-to-right, starting with the leftmost generator that appears when its argument is written, unless it reads a generator not obtained by applying $\sigma$to the last generator it read, at which point it stops.

Let $\varphi_\tau : C \to C$ be the monoid automorphism of C that permutes the generators accoring to the permutation $\tau \in S_n$. Prove that $\mathrm{Gal}(e_\sigma) = \{\varphi_\tau : \sigma\tau = \tau\sigma\}$.