Differentials. Let A and B be monoids, X a set, and $\cdot\colon X \times A \to X$ a right action of A on X. A *differential* on X taking values in B is a map $d\colon \{(x, xa)\colon x \in X \wedge a \in A\} \to B$ such that $d(x, x) = 1$ and $d(x, xa)d(xa, xaa') = d(x, xaa')$ for all $x \in X$ and $a, a' \in A$. NOTE: d is not necessarily a total map on $X \times X$.

Prop. Let X be a set, A and B be sectors, $\cdot$ a right action of A on X, and d a differential on X taking values in B. Then the map $e_x(a) = d(x, xa)$ is a calc.

Proof. We have $e_x(1) = d(x, x) = 1$ and

$$e_x(aa') = d(x, xaa') = d(x, xa)d(xa, xaa') = e_x(a)d(xa, xaa')$$

for all $x \in X$ and $a, a' \in A$.

Prop. Let A and B be sectors, $e\colon A \to B$ a calc, and let A act on itself on the right by multiplication. Then the map $d\colon \{(a, aa')\colon a, a' \in A\} \to B$ defined by $d(a, aa') = e^a(a')$ is a differential.

Proof. We have

$$d(a, aa')d(aa', aa'a'') = e^a(a')e^{aa'}(a'') = e^a(a'a'') = d(a, aa'a'')$$

and $d(a, a) = e^a(1) = 1$ for all $a, a', a'' \in A$.