Conjugation. Let C be a sector and $E = \{e\colon C \to C\colon e \text{ is a calc}\}$. Then, as we noted above, C acts on E on the right by shift, $e \cdot c = e^c = f$ where $f\colon C \to C$ is the unique (due to cancellation in the group that embeds C) map such that $e(c)f(c') = e(cc')$ for all $c, c' \in C$. In addition, $\mathrm{Aut}(C)$, the group of monoid automorphisms of C, acts on E on the right by conjugation, $e \cdot \varphi = e^\varphi = \varphi^{-1} \circ e \circ \varphi$. (In fact, something stronger holds, see below.) The proof follows by applying the calc property of e and homomorphism property of $\varphi \in \mathrm{Aut}(C)$,

$$\varphi^{-1}(e(\varphi(cc'))) = \varphi^{-1}(e(\varphi(c)\varphi(c'))) = \varphi^{-1}(e(\varphi(c))e^{\varphi(c)}(\varphi(c')))$$

$$= \varphi^{-1}(e(\varphi(c)))\varphi^{-1}(e^{\varphi(c)}(\varphi(c'))),$$

so if e is a calc, then $e \cdot \varphi$ is a calc, and we have

$$e \cdot (\varphi \circ \psi) = (\varphi \circ \psi)^{-1} \circ e \circ (\varphi \circ \psi) = \psi^{-1} \circ \varphi^{-1} \circ e \circ \varphi \circ \psi$$

$$= \psi^{-1} \circ (\varphi^{-1} \circ e \circ \varphi) \circ \psi = (e \cdot \varphi) \cdot \psi$$

and $e \cdot 1 = e$.

Note that for $F \subseteq E$ an arbitrary subset (even possibly empty) the Galois group

$$\mathrm{Gal}(F) = \{\varphi \in \mathrm{Aut}(C)\colon f^\varphi = f \,\forall f \in F\} = \cap_{f \in F} \mathrm{Aut}(C)_f$$

is the intersection of stabilizer subgroups of elements of F under conjugation of calcs.

Exercise. Adapt the proof above to show that for sectors C, D, the group $\mathrm{Aut}(C)$ acts on E on the right by $e \cdot \varphi(c) = e(\varphi(c))$ and $\mathrm{Aut}(D)$ acts on E on the left by $\psi \cdot e(c) = \psi(e(c))$.

Exercise. Let $C = \langle c_1, \ldots, c_n \rangle$ be the free group on n elements, $\sigma \in S_n$ a permutation, and $e_\sigma(c) = \mathrm{trunc}(c, \mathrm{iter}_\sigma(c))$. Every monoid automorphism $\varphi\colon C \to C$ arises from a permutation of the generators $\{c_i\}_{i=1}^n$, so let $\varphi_\sigma(c_i) = c_{\sigma i}$ and we have $\varphi_\sigma(\varphi_\tau(c_i)) = \varphi_\sigma(c_{\tau i}) = c_{\sigma\tau i} = \varphi_{\sigma\tau}(c_i)$, so $\sigma \mapsto \varphi_\sigma$ is a group homomorphism. Find $e_\sigma^{\varphi_\tau}$ for $\sigma, \tau \in S_n$.