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Electromagnetic concept of phase conjugation

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Abstract. The vectorial phase conjugation is examined using both the operation of complex conjugation applied to the vector complex amplitudes of the electromagnetic field and the temporal reversion principle. The temporal reversibility is discussed for the free electromagnetic field and the phase-conjugate field generated in the four-wave mixing.

1. Introduction

Except for applications, there are two classes of fundamental problems concerning optical phase conjugation. In the framework of the former class of problems, attention is focused on the phase conjugation as a possible property of the optical field itself. Under this concept, the process of generation of the phase-conjugate field is not taken into account so that the admissible states of the phase-conjugate field are verified by the homogeneous scalar wave equation in the framework of the scalar approximation or by the source-free Maxwell equations for the exact electromagnetic theory [1–3]. The latter class of problems is related to the process of generation of the phase-conjugate field [4–6].

In this paper, both the general properties and the generation of the phase-conjugate field are examined applying the exact electromagnetic theory. Although the optical phase conjugation is a well known phenomenon and it plays a very important role in optics, its general definition has not been formulated yet. The phase-conjugate field is usually presented as a temporal reversion of the given probe field. The term 'phase conjugation' has its origin in mathematics because the complex amplitudes of the phase-conjugate field are obtained owing to the operation of complex conjugation of the probe field. In the scalar approximation, both temporal reversion and complex conjugation are equivalent operations because they necessarily result in the same state of the field. In this paper, both the principles are applied to the exact vectorial electromagnetic theory. The temporal reversibility is verified to be a mathematical tool to obtain the electromagnetic field exhibiting an ability to retrace the given probe field. The existence of the process of temporal reversibility is discussed both for the free electromagnetic field and for its source realized by means of the four-wave mixing.

2. Phase conjugation of the optical field

2.1. *Scalar theory*

Let us consider scalar optical fields U_p and U_c whose complex amplitudes fulfil the scalar wave equation

$$\nabla^2 U_j - \frac{1}{v^2}\frac{\partial^2 U_j}{\partial t^2} = 0, \quad j = p, c, \tag{1}$$

and their energy flows can be expressed by vector $\mathbf{S}$ given by

$$\mathbf{S}_j = \frac{\partial U_j^*}{\partial t}\nabla U_j + \mathrm{cc}, \quad j = p, c. \tag{2}$$

In the framework of the scalar theory, the optical phase conjugation can be defined as a process which relates the complex amplitude of the phase-conjugate field U_c to that of the given probe field U_p by operation of the complex conjugation. The complex conjugation is necessarily accompanied by the temporal reversion and reversion of the energy flow, so that we can write

$$U_c(\mathbf{r}, -t) = qU_p^*(\mathbf{r}, t)$$

for complex conjugation and temporal reversion, and

$$\mathbf{S}_c(-t) = -|q|^2\mathbf{S}_p(t)$$

for reversion of the energy flow, where q is an arbitrary complex constant. The complex conjugation, temporal reversion and reversion of the energy flow are properties which equivalently define the scalar phase-conjugate field.

2.2. *Vectorial electromagnetic theory*

2.2.1. *Complex conjugation of the probe field.* Let us consider the electromagnetic probe field represented by the electric and magnetic strength vectors $\mathbf{E}_p$ and $\mathbf{H}_p$, which can be written as

$$\mathbf{E}_p = \bar{\mathbf{E}}_p + \mathrm{cc}, \tag{3}$$

$$\mathbf{H}_p = \bar{\mathbf{H}}_p + \mathrm{cc}, \tag{4}$$

where

$$\bar{\mathbf{E}}_p = \mathbf{e}_p \exp(\mathrm{i}\varphi), \tag{5}$$

$$\bar{\mathbf{H}}_p = \mathbf{h}_p \exp(\mathrm{i}\varphi), \tag{6}$$

and

$$\mathbf{e}_p = \int_{-\infty}^{\infty} \mathbf{a}_p(\omega) \exp(\mathrm{i}\omega t)\, \mathrm{d}\omega, \tag{7}$$

$$\mathbf{h}_p = \int_{-\infty}^{\infty} \mathbf{b}_p(\omega) \exp(\mathrm{i}\omega t)\, \mathrm{d}\omega. \tag{8}$$

In general, the monochromatic vector amplitudes $\mathbf{a}_p$ and $\mathbf{b}_p$ and the phase φ are functions of the space coordinates x, y and z. Owing to the Maxwell equations, they are coupled by the relation

$$\mathbf{b}_p = \frac{1}{\omega\mu}[(\mathbf{a}_p \times \nabla\varphi) + \mathrm{i}(\nabla \times \mathbf{a}_p)]. \tag{9}$$

The monochromatic amplitudes $\mathbf{a}_p$ and $\mathbf{b}_p$ are in general complex and they can become simultaneously real only under the plane-wave approximation. Owing to this, the phase of the realistic field cannot be uniform, that is phase terms of the vector field components are mutually different. A general definition of the phase conjugation based on the operation of complex conjugation can then be formulated as follows.

Statement 1: *The electromagnetic field is termed phase-conjugate field if components of its vector amplitudes are given as a linear superposition of the complex conjugate components of the vector amplitudes of the given probe field. Coefficients of the linear combination are represented by the matrices which can be in general non-diagonal.*

The process of phase conjugation then can be expressed as

$$(\bar{E}_c)_j = \sum_k \Gamma_{jk}(\bar{E}_p^*)_k, \quad j,k = x,y,z, \tag{10}$$

$$(\bar{H}_c)_j = \sum_k \Lambda_{jk}(\bar{H}_p^*)_k, \quad j,k = x,y,z. \tag{11}$$

The matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ describe amplitude and phase modulations which appear because of the process of phase conjugation. As the probe and phase-conjugate fields must fulfil Maxwell equations, the introduced matrices cannot be independent. For example, if both the matrices are diagonal, they are related as

$$\Lambda_{jj} = -\frac{[\Gamma \mathbf{a}_p^* \times \nabla\varphi - \mathrm{i}(\nabla \times \Gamma \mathbf{a}_p^*)]_{jj}}{[\mathbf{a}_p^* \times \nabla\varphi - \mathrm{i}(\nabla \times \mathbf{a}_p^*)]_{jj}}, \quad j = x,y,z. \tag{12}$$

2.2.2. *Temporal reversion of the probe field.* The scalar phase-conjugate field is usually considered to be a temporal reversion of the given probe field. Let us now examine the temporal reversibility of the vectorial electromagnetic field under certain restrictions concerning material properties.

Solution I (non-conducting medium): Taking into account the Maxwell equations and the constitutive relations

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}, \tag{13}$$

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \tag{14}$$

where $\mathbf{P}$ and $\mathbf{M}$ denote the electric and magnetic polarizations, the phase-conjugate field representing temporal reversion of the given probe field can be written as

$$\mathbf{E}_c(-t) = Q\mathbf{E}_p(t), \tag{15}$$

$$\mathbf{H}_c(-t) = -Q\mathbf{H}_p(t), \tag{16}$$

$$\mathbf{P}_c(-t) = Q\mathbf{P}_p(t), \tag{17}$$

$$\mathbf{M}_c(-t) = -Q\mathbf{M}_p(t), \tag{18}$$

$$\rho_c = Q\rho_p, \tag{19}$$

where ρ denotes the charge density and Q is an arbitrary real positive constant. The corresponding vector complex amplitudes can then be expressed in the form

$$\bar{\mathbf{E}}_c(-t) = Q\bar{\mathbf{E}}_p^*(t), \tag{20}$$

$$\bar{\mathbf{H}}_c(-t) = -Q\bar{\mathbf{H}}_p^*(t). \tag{21}$$

The current densities related to the phase-conjugate and probe field are required to fulfil the relation

$$\mathbf{j}_c(-t) = -Q\mathbf{j}_p(t). \tag{22}$$

Because of Ohm's law, equations (15) and (22) can be simultaneously fulfilled only in the non-conducting medium.

Solution II (non-conducting medium free from charge): Application of the temporal reversion to the Maxwell equations can also result in the solution which can be written as

$$\mathbf{E}_c(-t) = -Q\mathbf{E}_p(t), \tag{23}$$

$$\mathbf{H}_c(-t) = Q\mathbf{H}_p(t), \tag{24}$$

$$\mathbf{P}_c(-t) = -Q\mathbf{P}_p(t), \tag{25}$$

$$\mathbf{M}_c(-t) = Q\mathbf{M}_p(t). \tag{26}$$

The vector complex amplitudes are given as

$$\bar{\mathbf{E}}_c(-t) = -Q\bar{\mathbf{E}}_p^*(t), \tag{27}$$

$$\bar{\mathbf{H}}_c(-t) = Q\bar{\mathbf{H}}_p^*(t). \tag{28}$$

The current and charge densities are then required to fulfil the relations

$$\mathbf{j}_c(-t) = Q\mathbf{j}_p(t), \tag{29}$$

$$\rho_c = -Q\rho_p. \tag{30}$$

As is obvious, the last conditions can be fulfilled only in the non-conducting medium free from charge.

Solution III (homogeneous non-conducting medium free from charge): On the assumption that the medium is homogeneous (ϵ and μ are constant) and the conductivity and the charge density are equal to zero, the Maxwell equations provide a solution of the form

$$\mathbf{E}_c(-t) = Q\left(\frac{\mu}{\epsilon}\right)^{1/2}\mathbf{H}_p(t), \tag{31}$$

$$\mathbf{H}_c(-t) = Q\left(\frac{\epsilon}{\mu}\right)^{1/2}\mathbf{E}_p(t), \tag{32}$$

$$\mathbf{P}_c(-t) = Q\left(\frac{\epsilon}{\mu}\right)^{1/2}\mathbf{M}_p(t), \tag{33}$$

$$\mathbf{M}_c(-t) = Q\left(\frac{\mu}{\epsilon}\right)^{1/2}\mathbf{P}_p(t). \tag{34}$$

The vector complex amplitudes of the probe and phase-conjugate field are coupled by expressions

$$\bar{\mathbf{E}}_c(-t) = Q\left(\frac{\mu}{\epsilon}\right)^{1/2}\bar{\mathbf{H}}_p^*(t), \tag{35}$$

$$\bar{\mathbf{H}}_c(-t) = Q\left(\frac{\epsilon}{\mu}\right)^{1/2}\bar{\mathbf{E}}_p^*(t). \tag{36}$$

Owing to the restrictions concerning homogeneity of the medium, the last solution is of little importance because applications require realization of the phase conjugation in inhomogeneous media.

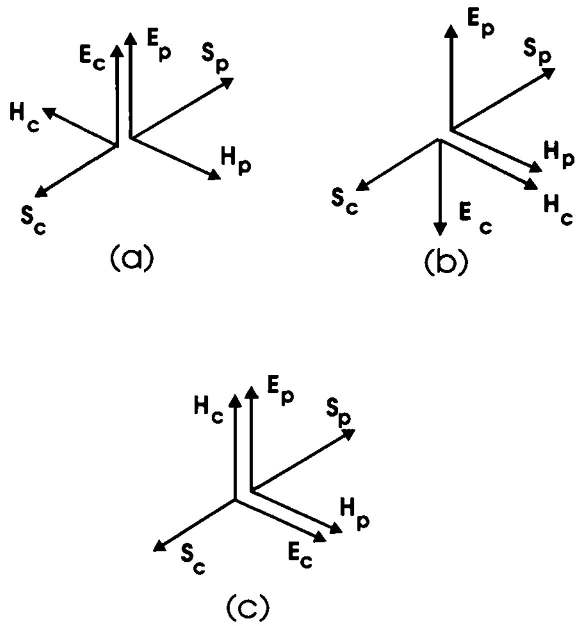

Figure 1. Electric and magnetic strength vectors and the Poynting vector for both the probe field and its time-reversed replica obtained by means of the homogeneous Maxwell equations: (*a*) solution I (non-conducting medium); (*b*) solution II (non-conducting medium free from charge); (*c*) solution III (homogeneous medium free from charge).

If the temporal reversibility is analysed by the Maxwell equations rewritten separately for $\bar{\mathbf{E}}_j$ and $\bar{\mathbf{H}}_j$, $j = p, c$, the more general relations between complex vector amplitudes of the phase-conjugate and probe field can be accepted. In this case, the real positive constant Q used in equations (20), (21), equations (27), (28) and equations (35), (36) can be replaced by a complex constant q in relation to the scalar description.

Solutions I–III are in agreement with the energy conservation law and they exhibit reversion of the electromagnetic energy flow given by the Poynting vector. At each point of the space, the Poynting vectors of the probe and phase conjugate fields have an opposite direction (figure 1) so that we can write

$$\mathbf{S}_c(-t) = -|q|^2 \mathbf{S}_p(t). \qquad (37)$$

Results following from the analysis of the temporal reversibility can now be summarized as follows.

Statement 2: *The process of the temporal reversion is not acceptable in conducting media. In non-conducting media free from charge, the homogeneous Maxwell equations admit two solutions (I and II) which are equivalent up to the phase factor* exp (iπ). *The direction of the Poynting vector of the phase-conjugate field defined by using the temporal reversion principle is at each point of the space exactly opposite with respect to that related to the probe field.*

Unlike the scalar theory, the field temporal reversibility is not directly related to the process of amplitude complex conjugation. The phase-conjugate field defined in terms of the complex conjugation (Statement 1) exhibits property of the reversion of the energy flow obtained owing to application of the temporal reversion principle (Statement 2) only if the matrices of the phase conjugation are restricted by additional conditions which can be written as

$$\Gamma_{jk} \mp q\delta_{jk} = 0, \quad j = x, y, z. \tag{38}$$

With respect to the condition (12), we can also write

$$\Lambda_{jk} \pm q\delta_{jk} = 0, \quad j = x, y, z. \tag{39}$$

Statement 3: *Definitions of the phase conjugation based on the complex conjugation operation and the temporal reversion principle become equivalent only if amplitude and phase modulations introduced during phase conjugation are uniform, that is if the matrices of the phase conjugation Γ and Λ can be replaced by the constants $\pm q$ and $\mp q$ respectively.*

3. Generation of the phase-conjugate field

In the previous section, the phase conjugation was examined as a source-free problem. It was verified that an ideal vector phase conjugation defined as the exact temporal reversion of the given electromagnetic field is admissible by the homogeneous Maxwell equations. In this section, the phase-conjugate field generation is examined.

We deal with four-wave mixing in Kerr-like media which is the well known interaction providing a good approximation of the phase conjugation. The interacting waves are described by the electric and magnetic strength vectors of the form

$$\mathbf{E}_j(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{E}}_j(\mathbf{r}, \omega_j) \exp(\mathrm{i}\omega_j t)\, \mathrm{d}\omega_j, \quad j = 1, 2, p, c, \tag{40}$$

$$\mathbf{H}_j(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{H}}_j(\mathbf{r}, \omega_j) \exp(\mathrm{i}\omega_j t)\, \mathrm{d}\omega_j, \quad j = 1, 2, p, c, \tag{41}$$

where the indices 1 and 2 denote pump waves and p and c are used for probe and phase-conjugate waves. We assume a collinear geometry with the boundary field intensities given by

$$\mathbf{E}_p(z = 0) \equiv \mathbf{E}_p^0, \tag{42}$$

$$\mathbf{H}_p(z = 0) \equiv \mathbf{H}_p^0, \tag{43}$$

$$\mathbf{E}_c(z = L) = 0, \tag{44}$$

$$\mathbf{H}_c(z = L) = 0. \tag{45}$$

The induced third-order electric polarizations related to the probe and phase-conjugate field can be written as

$$\mathbf{P}_j^{(3)}(\mathbf{r}, t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \bar{\mathbf{P}}_j^{(3)}(\mathbf{r}, \omega_j) \exp(\mathrm{i}\omega_j t)\, \mathrm{d}\omega_j, \quad j = p, c. \tag{46}$$

The monochromatic components of the nonlinear polarizations can be written in the form

$$\bar{\mathbf{P}}_j^{(3)}(\mathbf{r},\omega_j) = \epsilon_0 \frac{1}{(2\pi)^3}\int_0^\infty \mathrm{d}\omega_l \int_0^\infty \mathrm{d}\omega_m \int_0^\infty \mathrm{d}\omega_s\, \delta(\omega_j - \omega_l - \omega_m + \omega_s)$$

$$\times\, \chi^{(3)}(\omega_j)\vdots\bar{\mathbf{E}}_l(\mathbf{r},\omega_l)\bar{\mathbf{E}}_m(\mathbf{r},\omega_m)\bar{\mathbf{E}}_s^*(\mathbf{r},\omega_s), \quad \{j,l,m,s\} = \{1,2,p,c\}, \quad (47)$$

where

$$\bar{\mathbf{E}}_j^*(\mathbf{r},\omega_j) = \bar{\mathbf{E}}_j(\mathbf{r},-\omega) \quad (48)$$

$$\bar{\mathbf{P}}_j^{*(3)}(\mathbf{r},\omega_j) = \bar{\mathbf{P}}_j^{(3)}(\mathbf{r},-\omega_j), \quad (49)$$

and $\chi^{(3)}$ denotes the nonlinear susceptibility tensor. Under the assumption that the medium is isotropic, its components can be written as

$$\chi_{jlms}(\omega_i) = \chi_1(\omega_i)\delta_{jl}\delta_{ms} + \chi_2(\omega_i)\delta_{jm}\delta_{ls} + \chi_3(\omega_i)\delta_{js}\delta_{lm}. \quad (50)$$

Let us now examine conditions under which the phase-conjugate field generated in the interaction could represent a temporal reversion of the probe field. The general conditions valid for a non-conducting medium were discussed in the previous section and they are given by equations (15)–(19). Applying equations (40), (41) and (47), the conditions concerning electric and magnetic fields and nonlinear polarizations can be written in the form

$$\bar{\mathbf{E}}_c(\mathbf{r},\omega_c) = q\bar{\mathbf{E}}_p^*(\mathbf{r},\omega_p), \quad (51)$$

$$\bar{\mathbf{H}}_c(\mathbf{r},\omega_c) = -q\bar{\mathbf{H}}_p^*(\mathbf{r},\omega_p), \quad (52)$$

$$\bar{\mathbf{P}}_c^{(3)}(\mathbf{r},\omega_c) = q\bar{\mathbf{P}}_p^{*(3)}(\mathbf{r},\omega_p). \quad (53)$$

As is obvious, the monochromatic component of the source of the phase-conjugated field, that is the nonlinear polarization $\bar{\mathbf{P}}_c^{(3)}(\omega_c)$, must be related to that of the probe field by complex conjugation operation. With respect to equation (47), it can be achieved only if the following relation is valid:

$$\chi^{(3)}(\omega_c)\vdots\bar{\mathbf{E}}_1(\mathbf{r},\omega_1)\bar{\mathbf{E}}_2(\mathbf{r},\omega_2)\bar{\mathbf{E}}_p^*(\mathbf{r},\omega_p) = q^2\chi^{*(3)}(\omega_p)\vdots\bar{\mathbf{E}}_1^*(\mathbf{r},\omega_1)\bar{\mathbf{E}}_2^*(\mathbf{r},\omega_2)\bar{\mathbf{E}}_p^*(\mathbf{r},\omega_p). \quad (54)$$

Assuming that the interacting waves are transverse electromagnetic fields, the condition (54) can be rewritten in the component form as

$$\bar{P}_{c_j}^{(3)}(\mathbf{r},\omega_c) = q\bar{P}_{p_j}^{*(3)}(\mathbf{r},\omega_p), \quad j = x, y, \quad (55)$$

where the polarization components are given as

$$\bar{P}_{p_x}^{(3)}(\mathbf{r},\omega_p) = q[\kappa_{xxyy}^{(p)}\bar{E}_{p_x}(\mathbf{r},\omega_p) + \kappa_{xyyx}^{(p)}\bar{E}_{p_y}(\mathbf{r},\omega_p)], \quad (56)$$

$$\bar{P}_{p_y}^{(3)}(\mathbf{r},\omega_p) = q[\kappa_{yxxy}^{(p)}\bar{E}_{p_x}(\mathbf{r},\omega_p) + \kappa_{yyxx}^{(p)}\bar{E}_{p_y}(\mathbf{r},\omega_p)], \quad (57)$$

$$\bar{P}_{c_x}^{(3)}(\mathbf{r},\omega_c) = \kappa_{xxyy}^{(c)}\bar{E}_{p_x}^*(\mathbf{r},\omega_p) + \kappa_{xyyx}^{(c)}\bar{E}_{p_y}^*(\mathbf{r},\omega_p), \quad (58)$$

$$\bar{P}_{c_y}^{(3)}(\mathbf{r},\omega_c) = \kappa_{yxxy}^{(c)}\bar{E}_{p_x}^*(\mathbf{r},\omega_p) + \kappa_{yyxx}^{(c)}\bar{E}_{p_y}^*(\mathbf{r},\omega_p). \quad (59)$$

The introduced functions $\kappa_{jlms}^{(i)}$ describe coupling of monochromatic components of the interacting waves and they depend on the spatial structure and polarizations of pump waves:

$$\kappa^{(j)}_{xxyy} = \epsilon_0[\chi_0(\omega_j)\bar{E}_{1_x}(\mathbf{r},\omega_1)\bar{E}_{2_x}(\mathbf{r},\omega_2) + \chi_3(\omega_j)\bar{E}_{1_y}(\mathbf{r},\omega_1)\bar{E}_{2_y}(\mathbf{r},\omega_2)], \quad (60)$$

$$\kappa^{(j)}_{xyyx} = \epsilon_0[\chi_1(\omega_j)\bar{E}_{1_x}(\mathbf{r},\omega_1)\bar{E}_{2_y}(\mathbf{r},\omega_2) + \chi_2(\omega_j)\bar{E}_{1_y}(\mathbf{r},\omega_1)\bar{E}_{2_x}(\mathbf{r},\omega_2)], \quad (61)$$

$$\kappa^{(j)}_{yxxy} = \epsilon_0[\chi_1(\omega_j)\bar{E}_{1_y}(\mathbf{r},\omega_1)\bar{E}_{2_x}(\mathbf{r},\omega_2) + \chi_2(\omega_j)\bar{E}_{1_x}(\mathbf{r},\omega_1)\bar{E}_{2_y}(\mathbf{r},\omega_2)], \quad (62)$$

$$\kappa^{(j)}_{yyxx} = \epsilon_0[\chi_0(\omega_j)\bar{E}_{1_y}(\mathbf{r},\omega_1)\bar{E}_{2_y}(\mathbf{r},\omega_2) + \chi_3(\omega_j)\bar{E}_{1_x}(\mathbf{r},\omega_1)\bar{E}_{2_x}(\mathbf{r},\omega_2)], \quad (63)$$

where $\chi_0(\omega_j) = \chi_1(\omega_j) + \chi_2(\omega_j) + \chi_3(\omega_j)$, $j = p, c$. Applying equations (56)–(63), it can be verified that the following relations must be simultaneously valid to fulfil condition (55).

(i) $q = 1$,
(ii) $\chi_j(\omega_c) = \chi_j^*(\omega_p)$,
(iii) $\kappa^{(c)}_{xxyy} = \kappa^{*(p)}_{xxyy}$, $\kappa^{(c)}_{xyyx} = \kappa^{*(p)}_{xyyx}$,
(iv) $\kappa^{(c)}_{yxxy} = \kappa^{*(p)}_{yxxy}$, $\kappa^{(c)}_{yyxx} = \kappa^{*(p)}_{yyxx}$.

The relations (i)–(iv) show that the process of temporal reversion, alternatively defined by equation (55), can be achieved only in an approximate sense by four-wave mixing. The conditions (i) and (ii) show that neither amplification no attenuation is admissible in an ideal process of phase conjugation. Furthermore, the condition (ii) can be fulfilled only if the material dispersion is neglected. If such an assumption is taken into account, the condition (iii) can be interpreted as exact mutual phase conjugation of pump waves and the condition (iv) requires their linear polarization. Regardless of the conditions (ii)–(iv), the first condition itself shows that the ideal vector phase conjugation cannot be exactly realized. With respect to the boundary conditions, the coefficient q cannot be a unitary constant but it must be represented by a function of variable z which is equal to zero if $z = L$, that is $q(L) = 0$. Owing to this, we cannot deal further with the exact form of equations (51) and (52). Nevertheless, we can still assume that the best approximation to the temporal reversibility is achieved if equations (51) and (52) are fulfilled at the plane $z = 0$ of the mixing medium:

$$\bar{\mathbf{E}}_c(x,y,0,\omega_c) = q(0)\bar{\mathbf{E}}_p^*(x,y,0,\omega_p), \quad (64)$$

$$\bar{\mathbf{H}}_c(x,y,0,\omega_c) = -q(0)\bar{\mathbf{H}}_p^*(x,y,0,\omega_p). \quad (65)$$

The coefficient $q(0) \equiv q_0$ then represents the reflectivity coefficient of the phase-conjugate mirror and it can be obtained by solving the nonlinear wave equations describing the interaction. As is obvious even without solving the interaction, the electric and magnetic fields of the probe and phase-conjugate waves can be related by equations (64) and (65) only if the j component of the electric polarization depends only on the j component of the electric field:

$$\bar{P}^{(3)}_{c_j} \approx \bar{E}^*_{p_j}, \quad j = x, y, \quad (66)$$

$$\bar{P}^{(3)}_{p_j} \approx \bar{E}^*_{c_j}, \quad j = x, y. \quad (67)$$

Applying equations (56)–(63), it can be verified that this requirement can be fulfilled only if the probe and pump waves are linearly polarized in the same direction.

Up to now, the finite-frequency bandwidth of interacting waves has been considered. In real experiments, the nearly monochromatic interacting waves of the same frequency must be applied on account of the phase-matching condition. Taking into account such a degenerate interaction, the derived relations can be simplified owing to

$$\omega_1 = \omega_2 = \omega_p = \omega_c = \omega, \tag{68}$$

$$\kappa_{jmls}^{(p)} = \kappa_{jmls}^{(c)}, \tag{69}$$

$$\chi_1 = \chi_2. \tag{70}$$

Applying equations (56)–(63), it can be shown that the required forms of the electric polarizations (66) and (67) can be obtained for an arbitrary polarization of the probe wave if the pump waves are counter-rotating circularly polarized waves:

$$\bar{\mathbf{E}}_1 = \frac{1}{2^{1/2}}(E_{1_x}\mathbf{x} - \mathrm{i}E_{1_y}\mathbf{y}) \exp(\mathrm{i}kz), \tag{71}$$

$$\bar{\mathbf{E}}_2 = \frac{1}{2^{1/2}}(E_{2_x}\mathbf{x} + \mathrm{i}E_{2_y}\mathbf{y}) \exp(-\mathrm{i}kz), \tag{72}$$

where $\mathbf{x}$ and $\mathbf{y}$ denote the unit vectors. Such a conclusion is in agreement with results given in [7, 8].

3.1. *Example: electromagnetic plane-wave model of four-wave mixing*

The optical field generated in four-wave mixing best approximates a temporal reversion of the input probe field if all the interacting waves are linearly polarized in the same direction or if the pump waves have counter-rotating circular polarization. However, as was shown in the previous section, the exact temporal reversibility is impossible if the process of generation is taken into account. As the simplest example of this, let us now discuss a disturbance of the temporal reversibility process if the degenerate interaction of linearly polarized waves is examined.

We deal with the plane-wave electromagnetic theory of degenerate four-wave mixing assuming collinear geometry, undepleted pumping and linear polarization of interacting waves. Evolutions of the vector amplitudes of the probe and phase conjugate fields are driven by the coupled Helmholtz equations

$$(\nabla^2 + k^2)\bar{\mathbf{E}}_j = -\omega^2\mu\bar{\mathbf{P}}_j^{(3)}, \qquad j = p, c, \tag{73}$$

$$(\nabla^2 + k^2)\bar{\mathbf{H}}_j = -\mathrm{i}\omega^2\nabla \times \bar{\mathbf{P}}_j^{(3)}, \quad j = p, c. \tag{74}$$

The electric and magnetic fields of interacting beams are polarized along x and y directions respectively, so that we can write

$$(\bar{E}_j)_x = (e_j)_x \exp(\mathrm{i}k_j z), \quad j = 1, 2, p, c, \tag{75}$$

$$(\bar{H}_j)_y = (h_j)_y \exp(\mathrm{i}k_j z), \quad j = 1, 2, p, c, \tag{76}$$

where $k_j = -k$, $j = 1, p$, $k_j = k$, $j = 2, c$. Applying the slowly varying envelope approximation, we obtain the following set of coupled equations:

$$\frac{\mathrm{d}(e_p)_x}{\mathrm{d}z} = -\mathrm{i}\kappa(e_c^*)_x, \tag{77}$$

$$\frac{\mathrm{d}(e_c)_x}{\mathrm{d}z} = \mathrm{i}\kappa(e_p^*)_x, \tag{78}$$

$$\frac{\mathrm{d}(h_p)_y}{\mathrm{d}z} = -\mathrm{i}\frac{1}{\omega\mu}[|\kappa|^2(e_p^*)_x + \kappa k(e_c^*)_x], \tag{79}$$

$$\frac{\mathrm{d}(h_c)_y}{\mathrm{d}z} = -\mathrm{i}\frac{1}{\omega\mu}[|\kappa|^2(e_c^*)_x + \kappa k(e_p^*)_x], \tag{80}$$

where the coupling constant is given by

$$\kappa = \frac{k}{2}\chi_0(e_1)_x(e_2)_x. \tag{81}$$

Solutions obtained for the boundary conditions (42)–(45) can be expressed in the forms

$$(e_p)_x = \frac{\cos[|\kappa|(z-L)]}{\cos(|\kappa|L)}(e_p)_x^0, \tag{82}$$

$$(e_c)_x = \mathrm{i}\frac{\kappa}{|\kappa|}\frac{\sin[|\kappa|(z-L)]}{\cos(|\kappa|L)}(e_p^*)_x^0, \tag{83}$$

$$(h_p)_y = -\mathrm{i}\frac{1}{\omega\mu}\left(|\kappa|\frac{\sin[|\kappa|(z-L)]}{\cos(|\kappa|L)} + \mathrm{i}k\frac{\cos[|\kappa|(z-L)]}{\cos(|\kappa|L)}\right)(e_p)_x^0, \tag{84}$$

$$(h_c)_y = \frac{1}{\omega\mu}\left(\kappa\frac{\cos[|\kappa|(z-L)]}{\cos(|\kappa|L)} + \mathrm{i}k\frac{\kappa}{|\kappa|}\frac{\sin[|\kappa|(z-L)]}{\cos(|\kappa|L)}\right)(e_p^*)_x^0. \tag{85}$$

The reflectivity coefficients of the phase-conjugate mirror are related to the complex amplitudes of the electric and magnetic field as

$$R_e = \frac{(e_c)_x^0}{(e_p^*)_x^0}, \tag{86}$$

$$R_h = \frac{(h_c)_y^0}{(h_p^*)_y^0}, \tag{87}$$

and they can be expressed in the forms

$$R_e = -\mathrm{i}\frac{\kappa}{|\kappa|}\tan(|\kappa|L), \tag{88}$$

$$R_h = \mathrm{i}\frac{\kappa}{|\kappa|}\frac{(1-|\kappa|^2/k^2)\cos(|\kappa|L)\sin(|\kappa|L) + \mathrm{i}|\kappa|/k}{\cos^2(|\kappa|L) + (|\kappa|^2/k^2)\sin^2(|\kappa|L)}. \tag{89}$$

Accepting the concept of complex conjugation described by matrices $\mathbf{\Gamma}$ and $\mathbf{\Lambda}$ introduced in equations (10) and (11), the probe and phase-conjugate fields are related by the matrices given as

$$\Gamma_{ij} = \delta_{ij}\delta_{xj}R_e, \quad j = x, y, \tag{90}$$

$$\Lambda_{ij} = \delta_{ij}\delta_{yj}R_h, \quad j = x, y, \tag{91}$$

at the plane $z = 0$ of the mixing medium. If the generated field is required to be the best approximation to the temporal reversion of the probe field, both the fields must be related by equations (64) and (65), at the plane $z = 0$. This is possible only if the reflectivity coefficients are given by the real positive coefficient q_0 as

$$R_e = -R_h = q_0. \tag{92}$$

As is obvious, the reflectivity coefficients of the phase-conjugate mirror realized by means of the degenerate interaction of linearly polarized waves does not exactly fulfil condition (92). It can be fulfilled only in an approximate sense dealing with the assumption

$$\frac{|\kappa|}{k} \ll 1. \tag{93}$$

In practice, the condition (93) holds because the attainable intensities of pump waves and available nonlinear materials provide coupling constants which are negligible in comparison with the wavenumber.

4. Conclusion

In this paper, the existence of the electromagnetic field exhibiting an ability to retrace the given probe field is examined. The mathematical tools used to obtain such a phase-conjugate field are operations of the amplitude complex conjugation and the temporal reversion principle. While both the principles are equivalent in the scalar theory, they result in two possible definitions of the phase conjugation if the vectorial electromagnetic theory is applied (statements 1 and 2). The phase-conjugate electromagnetic field defined as a temporal reversion of the probe field has the Poynting vector which is at each point of the space exactly opposite to that of the probe field. Concerning the electromagnetic energy flow, the phase-conjugate wave exactly retraces the given probe wave. If the vectorial phase conjugation is defined by using the operation of amplitude complex conjugation (statement 1), the reversion of the Poynting vector is ensured only under certain restrictions given by statement 3. Thus the principle of the temporal reversion can be used to define process of ideal phase conjugation in both the scalar approximation and the exact electromagnetic theory. Analysis of the existence of the exact temporal reversibility of the electromagnetic field can be concluded as follows.

(1) Because of Ohm's law, the temporal reversibility of the Maxwell equations is not possible in a conducting medium.

(2) In a non-conducting medium free from charge there are two solutions of the temporal reversion of the homogeneous Maxwell equations. The solutions are identical up to a phase factor.

(3) An exact temporal reversibility (i.e. an ideal phase conjugation) is possible only for a free electromagnetic field. If the boundary and initial conditions related to a given source are taken into account, phase conjugation can be realized only in an approximate sense. For example, if a phase-conjugate field is generated in parametric degenerate four-wave mixing, the disturbance of the temporal reversibility is of the order of $|\kappa|/k$, where κ and k denote the coupling constant and wavenumber respectively.

The non-causality problem following from the temporal reversion [9] can be removed by using the retardation time t_0. The scalar amplitudes of the phase-conjugate and probe field are then related by $U_c(\mathbf{r}, t_0 - t) \approx U_p^*(\mathbf{r}, t)$. The concept of the retardation time is also applicable to the vectorial electromagnetic theory.

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