Proof of Barnett's Identity

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Abstract

In order for vortex equations on moduli spaces of Z_n orbifolds with $N = 1$ spacetime supersymmetry of hyper-Kahler Hirzebruch surfaces fibered over a lens space to be gauge mediated, one requires,

$$\sum_{k=0}^{\infty} k = \frac{e^{i\pi}}{11.999...}$$

This identity is proved via the thrice iterated application of integration operators, and by evaluating non-trivial zeros of real valued second degree polynomials of a single variable.

1 Proof

To begin with, we first want to prove that the sequence of partial sums (s_n) of the sum $\sum_{n\in\mathbb{N}} |(-1)^n \frac{x^{2n+1}}{(2n+1)!}|$ in $\mathbb{R}$ satisfies the property that $\exists$ some $x \in \mathbb{R}$ such that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N} :\ |s_n - x| < \varepsilon\ \forall n \geq \mathbb{N}$.
We have

$$\sum_{n\in\mathbb{N}} \left| (-1)^n \frac{x^{2n+1}}{(2n+1)!} \right| = \sum_{n\in\mathbb{N}} \frac{|x|^{2n+1}}{(2n+1)!}$$

But,

$$\sum_{n\in\mathbb{N}} \frac{|x|^{2n+1}}{(2n+1)!}$$

is just the terms of:

$$\sum_{n\in\mathbb{N}} \frac{|x|^{n}}{n!}$$

for odd n.

Thus,

$$\sum_{n\in\mathbb{N}} \frac{|x|^{2n+1}}{(2n+1)!} < \sum_{n=0}^{\infty} \frac{|x|^n}{n!}$$

However,

$$\sum_{n\in\mathbb{N}} \frac{|x|^n}{n!} = \exp|x|$$

which is from the Taylor series expansion of $e^{|x|}$, and the result follows from application of the Squeeze theorem.

The exact same argument can be used to show that $\sum_{n\in\mathbb{N}} (-1)^n \frac{x^{2n}}{(2n)!}$ also has the same properety that the sequence of partial sums (s_n) satisfy the condition that in $\mathbb{R}$, $\exists$ some $x \in \mathbb{R}$ such that $\forall \varepsilon > 0,\ \exists N \in \mathbb{N}:\ |s_n - x| < \varepsilon\ \forall n \geq \mathbb{N}$.

Now, when we have,

$$\sum_{n\in\mathbb{N}} (-1)^n \frac{\theta^{2n}}{(2n)!} + i \sum_{n\in\mathbb{N}} (-1)^n \frac{\theta^{2n+1}}{(2n+1)!}$$

we get an intriguing result. Since both sums are absolutely convergent they can be summed,

$$= \sum_{n\in\mathbb{N}} \left((-1)^n \frac{\theta^{2n}}{(2n)!} + i\,(-1)^n \frac{\theta^{2n+1}}{(2n+1)!} \right)$$

$$= \sum_{n\in\mathbb{N}} \left(\frac{(i\theta)^{2n}}{(2n)!} + \frac{(i\theta)^{2n+1}}{(2n+1)!} \right)$$

$$\sum_{n\in\mathbb{N}} \frac{(i\theta)^n}{n!}$$

$$= e^{i\theta}$$

Let $\theta = \pi$. Then since $\sum_{n\in\mathbb{N}} (-1)^n \dfrac{\theta^{2n}}{(2n)!} = -1$ and $\sum_{n\in\mathbb{N}} (-1)^n \dfrac{\theta^{2n+1}}{(2n+1)!} = 0$ we have ,

$$\sum_{n\in\mathbb{N}} (-1)^n \frac{\pi^{2n}}{(2n)!} + i \sum_{n\in\mathbb{N}} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!} + 1 = 0$$

For the second part of the proof, consider the set of rational numbers r such that $r < 0$, or $r < 0.9$, or $r < 0.99$, or r is less than some other number of the form $1 - (\frac{1}{10})^n$. This set is the number $0.999... \in \mathbb{R}$. Every element of 0.999... is less than 1, so it is an element of the real number 1. Conversely, an element of 1 is a rational number $\frac{a}{b} < 1$, which implies

$$\frac{a}{b} < 1 - (\frac{1}{10})^b$$

Since 0.999... and 1 contain the same rational numbers, they are the same set.

Finally, given what was deduced above and since,

$$e^{i\pi} = \sum_{n \in \mathbb{N}} (-1)^n \frac{\pi^{2n}}{(2n)!} + i \sum_{n \in \mathbb{N}} (-1)^n \frac{\pi^{2n+1}}{(2n+1)!}$$

Barnett's identity can thus be seen as an immediate collorary of Brady Haran's astounding result [1]. Thus,

$$\sum_{k=0}^{\infty} k = \frac{e^{i\pi}}{11.999...}$$

References

[1] B. Haran, "An Astounding Result", *Journal of Advanced Research in Pure Mathematics*. Series B, 2013.