The Requirements on the Non-trivial Roots of the Riemann Zeta via the Dirichlet Eta Sum

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Abstract

An explanation of the Riemann Hypothesis, using the well known Dirichlet Eta sum equivalence, is given in sections, beginning with a brief history of the paper and a statement of the problem. The next 3 sections dissect the complex Eta sum into 8 real valued sums and 2 constants. Parts 6 and 8 explain a recursive relationship between the sums and constants, via 2 systems of 2 equations, while parts 7 and 9 explain the conditions generated from both systems. Finally, section 10 concludes the explanation in terms of the original inputs of the Dirichlet Eta sum, proves Riemann's suspicion, and it shows that the only possible solution for the real portion of the complex input, commonly labeled a, is that it must equal 1/2 and only 1/2.

1 A Brief History of this Paper and its 2 Revisions

There are 3 versions of this paper, an original and 2 revisions, with this being the 2nd update and 3rd version overall. The paper was put into digital text format around September of 2019, and the first revision around April of 2021. The original contains all core elements of the proof, but suffers from 2 issues. The reasoning of why certain functions must equal 0 is clumsy, as I was using the logical connections I had made at the time, compared to subsequently realized ones with greater efficiency, and I was still understanding the deeper reasons to such. Secondly, the system of equations that arises in the later half of the proof was not thoroughly explained, was redundant in some ways, and the key connections within it were explained using ratios that didn't adequately shed light on the deeper logic of the system at the time.

By eighteen months later, I had gained more understanding of the system, and had made strides in explaining both the zeros in the first half of the proof and the system of equations in the second. The April 2021 revision made significant changes and improvements to the aforementioned issues, as well as to the editing, and to the clarity of the paper overall. However, and eventually, I thought the system had not yet revealed all of its secrets, nor had it been 100%

explained, and that at its core, it must still contain a clearer explanation. After recently solidifying my understanding of certain connections within the system, and after two and a half years of pondering later, there are enough compelling changes to warrant an update, one which I think represents the clearest core logic and reasoning of the system to date. As such, this revision of the paper was put into digital format as of December, 2023.

Briefly, the major changes to the paper are as follows. One, an even clearer reasoning is provided as to why certain values must be 0. And two, the entire system of equations in the later half is revisited in a more orderly manner, organized, context added, and additional redundancies found and removed. At the time of updating, I suspect this will likely be the last revision to the paper, and I feel I've now fully explained what was set out to be explained regarding this approach to the topic.

2 A Statement of the Problem and the General Approach to the Solution

The formal explanation begins with a well known version of the Riemann hypothesis based on the closely related Dirichlet Eta function. In that version, the Dirichlet Eta sum $\eta(s)$ is stated in a functional equation with the Riemann Zeta function $\zeta(s)$. This is done in order to analytically continue the domain of the Zeta function, and it is shown as equation 1.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \left(1 - 2^{1-s}\right)\zeta(s) \tag{1}$$

Using the Dirichlet Eta sum, the Riemann hypothesis is precisely stated as, "all the zeros of the Dirichlet eta function, falling in the critical strip $0 < \Re(s) < 1$, lie on the critical line $\Re(s) = 1/2$," where $\Re(s)$ is the real portion of the complex input s [1].That real portion is commonly labeled as a lower case letter A. What then is the nature of the zeros of the Dirichlet Eta function?

The Eta function is an infinite sum of fractions, sometimes totaling to zero, where the denominators of that fraction sequence are the changing index of the sum raised to a complex valued power s. Small s is a standard complex number given as a + bi. The numerator of the fraction within the sum also contains information. In this case, the numerator is a negative 1 raised to a power involving the index, and it causes the fraction to alternate between positive and negative. The goal then, and challenge of the hypothesis, is to explain why the value of a, in the domain between 0 and 1, must be 1/2, and only 1/2, in order for that entire infinite sum of fractions to sum to zero. This is stated as equation 2.

$$\eta(s) \equiv \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^{a+bi}} = 0 \tag{2}$$

With this starting point, the rest of the general approach is as stated in the paper's abstract. That is, the next 3 sections dissect the complex Eta sum into 8 real valued sums and 2 constants. Parts 6 and 8 explain a recursive relationship between the sums and constants, via 2 systems of 2 equations, while parts 7 and 9 explain the conditions generated from both systems. Finally, section 10 concludes the explanation in terms of the original inputs of the Dirichlet Eta sum, proves Riemann's suspicion, and it shows that the only possible solution for the real portion of the complex input, commonly labeled a, is that it must equal 1/2 and only 1/2.

The first major step after setting up the problem is separating the real and imaginary portions of the complex Eta sum. This is done such that there is no longer a complex number inside the single sum, but rather 2 real valued sums instead.

3 Separating the Real and Imaginary Portions of the Complex Sum

Start by using exponent rules on the index raised to a complex power, a + bi; equation 3.

$$n^s = n^{a+bi} = n^a n^{bi} \tag{3}$$

Then expand the complex exponent n^{bi} with Euler's well known formula. The result is shown in equation 4.

$$n^s = n^a \left(\cos\left(b \ln n\right) + i \sin\left(b \ln n\right)\right) \tag{4}$$

Put the now expanded form back into equation 2, and then express the numerator as a complex number, equation 5. Please also note that I changed the n-1 to n+1 out of personal preference of convention, as I had used it while working the problem out on paper. This is allowed, as it does not change any of the values of the fraction. That is, $(-1)^{n-1}$ will always equal $(-1)^{n+1}$ over the natural index.

$$\frac{(-1)^{n+1}}{n^s} = \frac{(-1)^{n+1} + 0i}{n^a \cos\left(b \ln n\right) + n^a \sin\left(b \ln n\right) i} \tag{5}$$

Next, use the general formula for dividing complex numbers, equation 6, to carry out the division shown in 7.

$$\frac{u + vi}{x + yi} = \frac{(ux + vy) + (vx - uy)\, i}{x^2 + y^2} \tag{6}$$

$$\begin{gathered} \frac{(-1)^{n+1} + 0i}{n^a \cos\left(b \ln n\right) + n^a \sin\left(b \ln n\right) i} = \\ \frac{(-1)^{n+1} n^a \cos\left(b \ln n\right)}{\left(n^a \cos\left(b \ln n\right)\right)^2 + \left(n^a \sin\left(b \ln n\right)\right)^2} + \frac{0 - (-1)^{n+1} n^a \sin\left(b \ln n\right)}{\left(n^a \cos\left(b \ln n\right)\right)^2 + \left(n^a \sin\left(b \ln n\right)\right)^2} i \end{gathered} \tag{7}$$

The result can be simplified by factoring out a n^a squared in the denominator, canceling one of them with the numerator, and by using trigonometry rules on the remaining sin squared plus cos squared to equal one. The complex input Dirichlet Eta sum can now be expressed as the sum-difference of 2 sums with only real inputs, equation 8.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\cos(b\ln n)}{n^a} - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(b\ln n)}{n^a} i = 0 \quad (8)$$

Notice that the left sum of the equivalent expression is real valued and deals with cosines, and that the right sum, though still sitting in front of the imaginary number i, is real valued in magnitude and deals with sines. Since the Dirichlet Eta sum is a sum of complex numbers, the result is also complex, which is expected. Therefore, in order for the original complex Eta sum to equal zero, and thus have a root, both the real and imaginary parts of its total must be zero. That is, it must be equal to 0 + 0i.

Factoring out and dividing away a constant -1 from equation 8 results in the 2 sums, equations 9 and 10, labeled A and B as follows.

$$A = \sum_{n=1}^{\infty} \frac{(-1)^{n}\cos(b\ln n)}{n^a} \quad (9)$$

A is also referred to as the real portion of the complex Dirichlet Eta sum.

$$B = \sum_{n=1}^{\infty} \frac{(-1)^{n}\sin(b\ln n)}{n^a} \quad (10)$$

B is also referred to as the imaginary portion of the complex Dirichlet Eta sum, though its magnitude is real valued.

The task now becomes to determine when these 2 new sums are both equal to zero at the same time. To do that, they will need to be broken down, and the first stage for such, is separating each of them into their even and odd parts.

4 Separating the Real and Imaginary Sums into their Even and Odd Portions

Instead of using one sum for each of A and B, as they are stated thus far, and instead of letting their indices n run over the full set of natural numbers, use 2 sums for each, separating the even and odd inputs of the indices. Do this by separating n into 2n, for the evens, and into 2n-1, for the odds. This is shown for both A and B in equations 11 and 12.

$$
\begin{aligned}
A = \sum_{n=1}^{\infty} \frac{(-1)^n \cos(b \ln n)}{n^a} = \\
\sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \cos(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \cos(b \ln 2n)}{(2n)^a} = 0
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
B = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(b \ln n)}{n^a} = \\
\sum_{n=1}^{\infty} \frac{(-1)^{2n-1} \sin(b \ln(2n-1))}{(2n-1)^a} + \sum_{n=1}^{\infty} \frac{(-1)^{2n} \sin(b \ln 2n)}{(2n)^a} = 0
\end{aligned} \tag{12}
$$

The behavior of -1 raised to even or odd powers allows the resulting sums of equations 11 and 12 to be simplified, obtaining 13 and 14 respectively.

$$A = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} = 0 \tag{13}$$

$$B = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2n)}{(2n)^a} - \sum_{n=1}^{\infty} \frac{\sin(b \ln(2n-1))}{(2n-1)^a} = 0 \tag{14}$$

The sums involving 2n are known as the even portions, and the sums with 2n-1, the odd portions. Notice in both cases, that it is the even sums minus the odd sums. Specifically labeling the 4 sums from equations 13 and 14 gives equations 15 through 18.

$$A_{even} = A_e = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2n)}{(2n)^a} \tag{15}$$

A_e is referred to as the real even portion.

$$A_{odd} = A_o = \sum_{n=1}^{\infty} \frac{\cos(b \ln(2n-1))}{(2n-1)^a} \tag{16}$$

A_o is referred to as the real odd portion.

$$B_{even} = B_e = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2n)}{(2n)^a} \tag{17}$$

B_e is referred to as the imaginary even portion.

$$B_{odd} = B_o = \sum_{n=1}^{\infty} \frac{\sin(b \ln(2n-1))}{(2n-1)^a} \tag{18}$$

B_o is referred to as the imaginary odd portion.

This isn't quite broken down enough yet, and in order to determine when these new sum-differences within the real and imaginary sums are equal to zero, they must be deconstructed further. However, the odd sums do not lend themselves to being broken down easily, if possibly at all. Luckily, the even sums do, and later, functional relationships for the odd sums will be found so that they can be substituted. In the mean time, the next main phase of the explanation requires separating the Sine and Cosine portions of the even parts.

5 Separating the Sin and Cos Portions of the Real Even and Imaginary Even Sums

To separate the even sums, begin with the ln(2n) using log rules, equation 19, and follow up with the trigonometry formulas for addition within Cosines and Sines, equations 20 and 21. The initial results are then shown in 22 and 23.

$$\ln 2n = \ln 2 + \ln n \tag{19}$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y \tag{20}$$

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \tag{21}$$

$$A_e = \sum_{n=1}^{\infty} \frac{\cos(b \ln 2)\cos(b \ln n) - \sin(b \ln 2)\sin(b \ln n)}{2^a n^a} \tag{22}$$

$$B_e = \sum_{n=1}^{\infty} \frac{\sin(b \ln 2)\cos(b \ln n) + \cos(b \ln 2)\sin(b \ln n)}{2^a n^a} \tag{23}$$

In this case, 22 and 23 have addition and subtraction over a common denominator, so they can each be separated into yet another 2 sums. Those resulting sums take the form of products of functions of a and b independent of the index, multiplied by a portion of the sum dependent on the index, and therefore, those independent portions that include a and b can be pulled out in front as constants. This is shown as equations 24 and 25.

$$A_e = \left(\frac{\cos(b \ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} \right) - \left(\frac{\sin(b \ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \right) \tag{24}$$

$$B_e = \left(\frac{\sin(b \ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} \right) + \left(\frac{\cos(b \ln 2)}{2^a} * \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \right) \tag{25}$$

Next, respectively label K_c and K_s for the new cosine based and sine based constants in equations 24 and 25, shown as 26 and 27.

$$K_c = \frac{\cos(b \ln 2)}{2^a} \tag{26}$$

$$K_s = \frac{\sin(b \ln 2)}{2^a} \tag{27}$$

Label the 2 different sums amongst 24 and 25, and note that the K constants are the same for the real even, A_e, and imaginary even, B_e, sums, only in different positions. These are equations 28 and 29.

$$C = \sum_{n=1}^{\infty} \frac{\cos(b \ln n)}{n^a} \tag{28}$$

This is known as the basic cosine sum.

$$S = \sum_{n=1}^{\infty} \frac{\sin(b \ln n)}{n^a} \tag{29}$$

This is known as the basic sine sum.

Now, finally, between the 10 terms A, B, A_e, A_o, B_e, B_o, K_c, K_s, C, and S, there is enough information to determine when the original infinite complex valued Dirichlet Eta sum is equal to zero, and to answer why the real variable lower case A within it must be equal to 1/2 and only 1/2. In order to do that, the next steps involve understanding what maintains an output of 0 throughout splitting the original Dirichlet Eta sum into 8 other sums and 2 constants.

6 The Recursive Functional Relationships Between the Sums

For the remaining sections, the indices and upper bounds of the sums do not change, and can be ignored, as they do not affect the relationships or outcomes.

The self referential relationships amongst the sums are generally stated in words as follows. The real and imaginary sums are broken into even and odd sums, and then the even sums are broken into sine and cosine sums. However, those new sine and cosine sums end up being composed in terms of the earlier even and odd parent sums, and thus create a dependency loop.

Stating the relation from equation 13, using labels 15 and 16, gives equation 30, which is the even and odd split of the real portion.

$$A = A_e - A_o \tag{30}$$

Likewise, stating the relation from 14, using labels 17 and 18, gives 31, which is the even and odd split of the imaginary portion.

$$B = B_e - B_o \tag{31}$$

With equation 8, it was noted that the sums A, eq. 9, and B, eq. 10, must both be 0, and this is stated again with eq. 30 and eq. 31 as requirements in the equations in 32.

$$A = A_e - A_o = 0 \quad \text{AND} \quad B = B_e - B_o = 0 \tag{32}$$

This leads to the requirements in eq. 33.

$$A_e = A_o \quad \text{AND} \quad B_e = B_o \tag{33}$$

Using the labels from equations 26-29, equations 24 and 25 are written as 34 and 35.

$$A_e = K_c C - K_s S \tag{34}$$

$$B_e = K_s C + K_c S \tag{35}$$

Now review and more closely examine equations 28 and 29. Do the cosine and sine sums look familiar? They sure look like the real sum A, eq. 9, and the imaginary sum B, eq. 10, except for the -1 raised to the power, that is, except for the alternating part. In fact though, that is exactly what they are! The alternating real and imaginary sums, eqs. 9 and 10, subtract out every other term, while the sine and cosine sums, eqs. 28 and 29, add all the terms, of an otherwise identical sum. What are those other terms, which are being subtracted in the case of the real and imaginary sums, but are being added in the case of the sine and cosine sums? Equations 13 and 14 show that those terms turn out to be the odd function sums! That is, the real and imaginary sums are the difference of their respective even and odd sums, while the cosine and sine sums are the sum of their respective even and odd sums. This gives equations 36 and 37.

$$C = A_e + A_o \tag{36}$$

$$S = B_e + B_o \tag{37}$$

Adding 2 copies of the corresponding odd function to each side of the equations in 32, using A and B in terms of the difference of even and odd sums, and then substituting with equations 36 and 37 respectively, gives the 2 sets of equations shown in 38. This is the same as using eq. 33 with eqs. 36 and 37.

$$C = A_e + A_o = 2A_o \quad \text{AND} \quad S = B_e + B_o = 2B_o \tag{38}$$

Eq. 33 requires $A_e = A_o$ and $B_e = B_o$, such that C and S can be expressed as 2 even sums or 2 odd sums, as shown in eq. 39.

$$C = 2A_e = 2A_o \quad \text{AND} \quad S = 2B_e = 2B_o \tag{39}$$

It is now possible to fully state the system, to understand the requirements on the sums and constants within it, to explore its solutions, and to examine the implications of such.

7 The Conditions Generated by the Primary System of Sums and Constants

At a minimum, eq.33 shows that the corresponding even and odd sums, for A and B respectively, must have the same value. Consider equation 33 by splitting it into 2 cases. Let case 1 be the sums having a shared value of 0, and let case 2 be sharing any value other than 0. It is now shown by way of contradiction that it must be case one, that the shared value is 0, and furthermore, that it leads to the requirement that all the sums must individually be 0 at the same time.

7.1 Basic Simplification and the Quadratic of the System

Basic simplification of the system, in terms of the minimum number of variables, shows a separable relationship between the constants and the sums. It reveals a product of 0 relationship between a quadratic involving the system constants, and the sums. This can be done in terms of either the C or S sum, and to that end, C is used for this explanation.

Using eq.39, and substituting into 34 and 35, gives the system of equations in 40 and 41.

$$C = 2\left(K_c C - K_s S\right) \tag{40}$$

$$S = 2\left(K_s C + K_c S\right) \tag{41}$$

Solving for S in eq.40 gives the following.

$$S = \frac{\left(K_c - \frac{1}{2}\right)}{K_s} C \tag{42}$$

Substituting 42 into 41, and simplifying, leaves 43.

$$\left(K_c^2 - K_c + K_s^2 + \frac{1}{4}\right) C = 0 \tag{43}$$

This shows that either C is 0, the portion in the parentheses is 0, or both parts are 0. In any case where C is 0, by itself or with the quadratic, it means from eq.39 and eq.42 that case 1 must be true. Using the quadratic equation on the parentheses gives eq.44.

$$K_c = \frac{1 \pm \sqrt{-4K_s^2}}{2} \tag{44}$$

From eq.27, and the problem in general, it is known that K_s is real valued, and therefore its square will be positive. Similarly, from eq.26, K_c is real valued. Because of the -4 inside the square root, the only possible solution is for $K_s = 0$ which makes $K_c = 1/2$. This creates a contradiction as follows.

If $K_s = 0$, then eq.27 requires that $b \ln 2 = n\pi$, a multiple of pi, for some generic integer n. However if that's the case, and $b \ln 2 = n\pi$, then the numerator of eq.26 is plus or minus 1, and compared to the results just obtained from eq.44, you get the following.

$$\frac{1}{2} = K_c = \frac{\pm 1}{2^a} \tag{45}$$

This then requires that a is either complex valued or equal to 1, which places it outside the original domain of a, and it means that the solutions within the domain of a, of which we know there are at least some solutions when a=1/2, occur in the case when C is 0.

7.2 Requirements on the Sums of the System

A simple rearranging of equations 40 and 41 gives equations 46 and 47.

$$K_c - K_s \left(\frac{S}{C}\right) = \frac{1}{2} \tag{46}$$

$$K_c + K_s \left(\frac{C}{S}\right) = \frac{1}{2} \tag{47}$$

Observing the forms of eq.46 and eq.47, and setting them equal, gives eq.48, and after cross multiplying, eq.49. This shows that in order for the system to hold, as far as the sums are concerned, that the ratios of the basic sine and cosine sums must be negative reciprocals of each other. Since the values of the sums are real, it also shows that the only solution is for the sums to equal 0, and thus coincides with the result from the contradiction found within the quadratic in section 7.1.

$$-\frac{S}{C} = \frac{C}{S} \tag{48}$$

$$-S^2 = C^2 \tag{49}$$

The combined results from sections 7.1, 7.2, and equations 30, 31, and 39, are summed up in the very general eq.50, stating that all the sums are 0 when the system falls into equilibrium.

$$A = B = A_e = A_o = B_e = B_o = C = S = 0 \tag{50}$$

7.3 Requirements on the Constants of the System

Another rearrangement of 40 and 41, now also 46 and 47, gives equations 51 and 52.

$$\frac{S}{C} = \frac{2K_c - 1}{2K_s} \tag{51}$$

$$\frac{S}{C} = \frac{2K_s}{1 - 2K_c} \tag{52}$$

Observing the forms of eq.51 and eq.52, and setting them equal, gives eq.53, and after cross multiplying, eq.54. This shows that in order for the system to hold, as far as the constants are concerned, that equation 54 must be true.

$$\frac{2K_c - 1}{2K_s} = \frac{2K_s}{1 - 2K_c} \tag{53}$$

$$-4K_c^2 + 4K_c - 1 = 4K_s^2 \tag{54}$$

7.4 A Brief Review of the System and the Dependency of Constants

At this point in the explanation, I find it helpful to briefly review and summarize certain specific and overarching structures of the system. The equations at hand show how the components a and b independently control both the constants K_c and K_s, as well as the basic sine and cosine sums, and it is known that the system has solutions when the basic sums are 0, or when the quadratic is honored, and that we are looking for the values of a and b which satisfy such. However, it's also important to note and reiterate, that due to the nature of of the sine, cosine, and natural logs within the numerators of equations 26, 27, and their interactions with either eq.43 or eq.54, that neither K_c or K_s can take a value of 0, as it instantly removes all solutions for a within the domain. And yet, the only current solutions to the quadratic appear to be 0 and 1/2, so what gives?

With the sums necessarily being 0, and the quadratic seemingly unsolvable within the domain, the final piece of the puzzle is to realize that the system constants K_c and K_s already share a relationship to each other via the inputs a and b. The goal of this realization is then that the dependency between the system constants will change and collapse the quadratic into something with solutions within the domain once the substitution is made. Dividing eq.27 by eq.26 gives a quick, but less useful in this case, equation for this relationship. It shows that the ratio of the system constants only depends on b, making the ratio completely independent of the input a, and that it is proportional to the tangent of the natural log. However, that version does not immediately help with simplification, and another version of the relationship must be found within the information.

8 A Second System Using the Odd Sums

A second system of equations can now be built from the real and imaginary odd sums.

8.1 Sums and Differences of 2 Values which are Zero

The observation that if 2 separate real values are both 0, then the sum and difference of those 2 values are also both 0, is indeed a relatively trivial and elementary recognition. However, this is exactly what is needed to find another representation of the relationship between the system constants, and it turns out to be a powerful tool in finishing the problem.

Using the fact that the sums are each 0, combined with the sum and difference tool, we now return to the odd sums and use them to create a second system of equations.

8.2 The Odd Sum System

Using the sum and difference of the Odd sums, along with substitutions from eq.38, gives the following requirements.

$$A_o + B_o = C - A_e + S - B_e = 0 \quad \text{AND} \quad A_o - B_o = C - A_e - S + B_e = 0 \tag{55}$$

Substitute within eq.55, again using equations 34 and 35 for the even sums as before. This gives 56 and 57.

$$C - K_c C + K_s S + S - K_s C - K_c S = 0 \tag{56}$$

$$C - K_c C + K_s S - S + K_s C + K_c S = 0 \tag{57}$$

Solving for C in 56 gives eq.58.

$$C = \frac{(K_c - K_s - 1)}{(-K_c - K_s + 1)} S \tag{58}$$

Plugging 58 into 57, and then simplifying, gives 59.

$$(2K_c^2 - 4K_c + 2K_s^2 + 2)S = 0 \tag{59}$$

Here again we see the product of 0 relationship between a quadratic of system constants and the sum, though this time it was solved against the S sum. Rearranging eq.56 and eq.57 to find the new requirements on the constants within the odd system, as was done in section 7.3, gives equations 60 and 61.

$$\frac{S}{C} = \frac{(K_s + K_c - 1)}{(K_s - K_c + 1)} \tag{60}$$

$$\frac{S}{C} = \frac{(K_c - K_s - 1)}{(K_s + K_c - 1)} \tag{61}$$

Equating eq.60 to eq.61 and cross multiplying gives eq.62.

$$-2K_c^2 + 4K_c - 2 = 2K_s^2 \tag{62}$$

This provides a second equation showing the connection between constants based strictly on the constants, one without cumbersome trig terms. It can now be combined with equation 54 to determine the final requirements on the inputs a and b within system.

9 Combining the Requirements on the Constants

At this point, there are 2 quadratic equations with 2 unknowns, which govern the relationship between the sine and cosine constants, K_s and K_c, within the overall system. The first equation maintains the validity of the primary system of equations, independent of the values of the sums. The second equation is a result of the fact that the odd sums are 0, which further governs the constraints on the constants. Together, they now restrict and show that there is a very specific value dependent on K_s and K_c, but which is completely independent from the original inputs a and b. In fact, it is this very value which in turn governs a, the real portion of the complex input.

9.1 The Sum of the Squares of the Constants

For convenience, eq.54 and eq.62 are provided again below.

$$-4K_c^2 + 4K_c - 1 = 4K_s^2 \tag{63}$$

$$-2K_c^2 + 4K_c - 2 = 2K_s^2 \tag{64}$$

Combining the equations by subtracting one from the other and simplifying, removes the base K_c terms, and it leaves a single valued relationship between the squares of the constants.

$$K_c^2 + K_s^2 = \frac{1}{2} \tag{65}$$

This is rather interesting in that no matter the initial inputs a and b, the sum of the squares of the system constants always equals 1/2 when there are roots, and it means that it must be 1/2 if one wants to simultaneously make both the odd sums 0 and generate roots for the primary system. From this requirement, one can quickly show why the value of a must be 1/2.

9.2 The Value of the Real Portion of the Complex Input

Now, directly consider the sum of the squares of the constants using their full forms from eqs. 26 and 27.

$$K_c^2 + K_s^2 = \frac{\cos^2(b \ln 2)}{2^{2a}} + \frac{\sin^2(b \ln 2)}{2^{2a}} = \frac{1}{2^{2a}} \tag{66}$$

Simplifying the center of eq.66, with the familiar trigonometric identity, condenses the sine and cosine squares to 1, and it leaves the right side of the equation. This shows that the value of the sum of the squares of the constants is independent of the imaginary input b, which technically we already knew from approaching from the system side of things, and that coming from this side of things, it only relies on the real input lower case a. However, we just finished showing that the value of sums of squares is fixed, eq.65, which then leads to eq.67.

$$\frac{1}{2^{2a}} = \frac{1}{2} \tag{67}$$

10 Conclusion

Eureka, finally, there it is! Eq.67 leaves $2a = 1$, and therefore eq.69, a equals 1/2 and only 1/2!

$$2a = 1 \tag{68}$$

$$a = \frac{1}{2} \tag{69}$$

This shows that there is only one possible choice for the real value of the complex input which allows the recursive systems of sums and coefficients to balance, such that all 8 sums, and thus the original Dirichlet Eta function, are equal to 0 and generate roots.

Therefore, a must = 1/2, and Riemann's suspicions were correct!

I hope you enjoyed the explanation.

Q.E.D.

References

[1] Goodman, Len and Weisstein, Eric W., *:"Riemann Hypothesis."*, From MathWorld–A Wolfram Web Resource., https://mathworld.wolfram.com/RiemannHypothesis.html, 1999-2021.