Lorentz Transformations in the Modified Cosmological Model

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Abstract

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§1 Purpose and Scope

In [1], Steane says that there are two ways of thinking about a spinor: a two-component vector with complex entries, or a null 4-vector with real entries. However, we want to examine what happens if we convert to the Euclidean metric with complex 4-vectors. Also, if $\chi^4_\pm$ is oppositely timelike and spacelike in $\Sigma^\pm$, then when we assemble 4-vectors from x^i and either of $x^0, \chi^4_\pm$, there are only null vectors in one variant of $\chi^4_\pm$. Therefore, we need to work out all of the mechanics for the conventions on real and complex 4-vectors. Carroll's treatment of 4-vectors in spacetime in [2] requires real-valued 4-vectors, so we will examine the details.

Spinors have different x-form properites under Lorentz transformations, so we need to examine what they are.

The Lorentz invariant associated with a 4-vector x^μ (where the sum $x^\mu \hat{e}_\mu$ is implied) is the inner product $x^\mu x_\mu$. This is called the "spacetime interval" or the "Minkowski length squared." (CHECK ON SQUARED???) For a 4-vector in $\mathbb{R}^4$,

$$x^\mu = (t, x, y, z) \quad ,$$

the inner product is written

$$\big(x^\mu\big)^2 = x^\mu x_\mu = x^\mu \big(g_{\mu\nu} x^\nu\big) \quad .$$

We have the freedom to write the metric in one of two sign conventions:

$$\eta_{\mu\nu} = \mathrm{diag}(\pm 1, \mp 1, \mp 1, \mp 1)$$

so the inner product becomes

$$\begin{aligned} \big(x^\mu\big)^2 &= (t, x, y, z) \cdot (\pm t, \mp x, \mp y, \mp z) \\ &= \pm \big(t\big)^2 \mp \sum_i \big(x^i\big)^2 \end{aligned}$$

As these are oppositely signed, we will have a null interval.

§2 Main Results

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§3 The Manifold of Special Relativity

An interesting result in group theory is that certain groups have an associated manifold. The Lorentz group is such a group, and the associated manifold is what physicists call Minkowski space, which is the manifold of special relativity. Unlike curved spaces in the general theory of relativity, the special theory is confined to flat Lorentzian 4-space: Minkowski space. The flat Lorentzian metric (called the Minkowski metric) may be chosen in signature $\{+ - --\}$ or $\{- + ++\}$ so that the line element is

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \quad , \quad \text{or} \qquad ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \quad ,$$

and this is reflected in the convention to label the Lorentz group as O(3,1) or O(1,3).[1] The points in the associated manifold are specified with three real numbers and one imaginary, or three imaginary and one real. However, the current trend in physics is to label points in Minkowski space with $x^\mu \in \mathbb{R}^4$, such and then obtain the manifold of special relativity by defining the distance between special relativistic events with the flat Lorentzian metric

$$\eta_{\mu\nu} = \begin{bmatrix} \pm c^2 & 0 & 0 & 0 \\ 0 & \mp & 0 & 0 \\ 0 & 0 & \mp & 0 \\ 0 & 0 & 0 & \mp 1 \end{bmatrix} \qquad \Longrightarrow \qquad ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu \quad .$$

Since $\mathbb{R}^4$ is the manifold of O(4), we must associate vectors in $\mathbb{R}^4$ with the Lorentz group by introducing complexity with a non-Euclidean metric.

The main point of this paper will be to examine the natural representation of the Lorentz group: three real numbers and one imaginary, or vice versa. In the vector space of the natural Lorentz manifold, call it M with definition

$$M = \begin{cases} i\mathbb{R} \times \mathbb{R}^3 & \text{if} \quad \text{O(3,1)} \\ \mathbb{R} \times i\mathbb{R}^3 & \text{if} \quad \text{O(1,3)} \end{cases} \qquad ,$$

the metric will be the Euclidean one. As transformations on an other-than-usual vector space, the elements of the Lorentz group, the Lorentz transformations themselves, will not have their usual analytical forms when written as matrices. Since $\mathbb{R}^4$ and $\mathcal{H}$ are diffeomorphic vector spaces, but the manifolds associated with matrix groups O(3,1) and O(4) are not diffeomorphic, we need to study the differences. First, we need to establish a formal system definitions for all of this. We will review the basic physics described by 4-vectors in Minkowski space, then we will exceed vector analysis with the introduction of Riemannian geometry and establish a context in the Modified Cosmological Model.

[1] One often distinguishes the proper and restricted Lorentz groups from the full Lorentz group so that the freedom to choose a sign in the metric is reflected in a choice between SO(3,1) and SO(1,3) or $\text{SO}_+(3,1)$ or $\text{SO}_+(3,1)$.

§3.1 Brief Synopsis of Special Relativity

Since observers in different reference frames ought to be able to agree on the facts of objective phenomena in their joint experience, we are able to obtain tight constraints on what mathematical forms our physical theories might take. For two frames to be different—as opposed to the case of two different observers observing things in the same frame—one frame must be in motion *relative* to the other, and we arrive at the concept of *relativity*. Of particular interest is the case in which the origins of coordinates in each reference frame are separated by a time-varying displacement vector: one frame is moving with velocity $\vec{v}(t)$ relative to the other. To simplify things, we often consider the case when $\vec{v}$ is constant, and relative orientation of the coordinate axes in each reference frame is fixed; there is no acceleration or rotation between reference frames: one is simply and steadily moving relative to the other. In the so-called *standard configuration*, we choose things to be such that the two coordinate systems were at the same place at $t = 0$. Since $\vec{v}$ is a constant, we will define the coordinates of 3-space to be such that $\vec{v} = v\hat{x}$: the constant velocity is purely in the x-direction. Taking frame S with coordinates (t, x, y, z) as fixed while frame S' with coordinates (t', x', y', z') moves along the x-axis, the coordinates of an event with a time and place observed from S' might be described as

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \quad . \end{aligned}$$

If a bird is flying away from S with the same speed and direction as S', then its x position at time t is, from Newtonian dynamics, $x(t) = vt$, and we obtain $x'(t') = 0$. The bird remains stationary at the origin in S' for any value of t' because it is co-moving with S'. This formulation reflects what is called Galilean relativity. It is a good approximation at low speeds, which are called *non-relativistic*. However, if the bird becomes a photon, then it will be moving at the speed of light c relative to both S and S', and it cannot possibly be stationary in either frame. Clearly, the Galilean theory of relativity will not comply with the requirement that different observers' theories must agree on objective facts. A *special* theory of relativity is needed when the bird flies very fast. Loosely following Tipler and Llewellyn [3], we may derive the correct coordinate transformations as follows.

Assume the special relativistic transformation of position is

$$x' = \gamma\big(x - vt\big) \quad , \quad \text{where} \qquad \gamma = \gamma(v) \quad .$$

This will reduce to the Galilean expression as $v \to 0$, and it will yield the correct transformation otherwise. Since our theory must be consistent if S' is moving with speed v relative to S or if S is moving with speed $-v$ relative to stationary frame S', we obtain the equivalent expression

$$x = \gamma\big(x' + vt'\big) \quad .$$

By inserting the expression for x' into the one for x, we obtain

$$t' = \gamma\left[\frac{x\big(1-\gamma^2\big)}{\gamma^2 v} + t\right] \quad .$$

Since our bird has become a photon moving with velocity c in both the S and S' frames (irrespective of the magnitude of their relative velocities!), the position of the photon will be described as

$$x(t) = ct \quad , \qquad \text{and} \qquad x'(t') = ct' \quad .$$

Form the former, we obtain the constraint $x/t = c$, and into the latter we insert our expressions for x' and t':

$$\gamma\big(x - vt\big) = c\gamma\left[\frac{x\big(1-\gamma^2\big)}{\gamma^2 v} + t\right] \quad .$$

Multiplying by $\gamma v t^{-1}$ and using $x/t = c$ yields

$$\gamma^2 v\big(c - v\big) = c^2\big(1-\gamma^2\big) + \gamma^2 vc \quad .$$

Canceling like terms and dividing by c^2 yields an expression which is easily solved to obtain

$$\gamma = \frac{1}{\sqrt{1-\big(\frac{v}{c}\big)^2}}$$

Since Lorentz had derived this transformation before Einstein found its application in the physics of simultaneity, the resultant expressions

$$x' = \gamma\big(x - vt\big) \quad , \qquad \text{and} \qquad t' = \gamma\left(t - \frac{vx}{c^2}\right) \quad ,$$

reflect what is called “a rotation-free Lorentz transformation.” More concisely, such transformations are called *boosts*. Once rotations are added and we allow arbitrarily directed boost velocities $\vec{v} = v\hat{n}$, and once we add things like parity and time-reversal, the set of all such transformations forms “the Lorentz group.” Rather than the general group properties, the scope of the present work is confined to a particular representation of the group in the MCM. Still, the reader is referred to Appendix 2 in Roman [4] for an excellent and concise, 30-page statement of everything 90% of physicists will ever need to know about group theory.

Definition 3.1.1 The Lorentz factor is

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \quad , \qquad \text{where} \qquad \beta = \frac{v}{c}$$

Definition 3.1.2 A Lorentz frame, or an inertial frame, is KKKKKKKKKKKKKKKKK

• For a Lorentz frame, or a locally inertial frame, to be physical, we need to take the limit as the radius of the frame goes to zero.

Example 3.1.3 Spinning bucket. If one frame is as good as another, why does the water go up the sides of the bucket in the frame co-rotating with the bucket? We can rule out that the background of the universe's gravitational mass is dragging the water up the sides because the distance is to even the nearest celestial background object is not causally separated from the water going up the sides. However, if we assign a gravitational component to the infinite energy of the quantum vacuum, then we are able to achieve the requisite proximity needed to explain the immediate response of the water as dragging against a background.

Definition 3.1.4 An event is something that happens at a time and place. Events are uniquely specified with 4-vectors whose first component is a time t and whose other three components as a vector $\vec{x}$ in Euclidean 3-space.

Definition 3.1.5 The Minkowski metric is a rank-(0,2) tensor

$$\eta_{\mu\nu} = \text{diag}(\pm c^2, \mp 1, \mp 1, \mp 1) \quad .$$

Definition 3.1.6 The Minkowski square of a 4-vector x^μ is its double contraction with the metric $g_{\mu\nu}$.

Remark 3.1.7 In the usual conventions for special relativity the 4-vector in question is $x^\mu \in \mathbb{R}^4$, and the metric is the Minkowski metric $\eta_{\mu\nu}$ so

$$x_M^2 = \eta_{\mu\nu} x^\mu x^\nu = \pm c^2 \big(x^0\big)^2 \mp \Big[\big(x^1\big)^2 + \big(x^2\big)^2 + \big(x^3\big)^2 \Big] \quad .$$

However, this will not be the formula for x_M^2 when $x^\mu \in \mathcal{H}$. Such distinctions are made in Section 3.3.

Definition 3.1.8 A dimensional transposing parameter is a dimensionless scalar constant that changes the units of one quantity to another by multiplication. The speed of light c is the dimensional transposing parameter between time and space:

$$\frac{[\text{length}]}{[\text{time}]} \times [\text{time}] = [\text{length}] \quad .$$

Remark 3.1.9 It has not been established that the MCM's chronological and chirological times have the same units, so we may need to introduce new transposing parameters for $\chi^4_\pm$. Since it is expected that the relative scale of physics changes across sequential MCM worldsheets, the mechanism by which c exchanges small amounts of time for large amounts of space may have further utilities toward the changing the *level of aleph* [5].

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MAKE FIG Minkowski space
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Definition 3.1.10 The spacetime interval Δs^2 between two events in Minkowski space is the Minkowski square of their separation vector.

Definition 3.1.11 If the spacetime interval between two events vanishes, $\Delta s^2 = 0$, then the events are said to be *lightlike* separated, separated by the null interval, or separated by the lightlike interval. The separation vector between such events is said to be on *the light cone*, which is the union of all lightlike separation vectors. A 4-vector is lightlike if and only if its Minkowski square vanishes: $x_M^2 = 0$. Lightlike separation between two events is a type of *causal* separation because correlations restricted by the speed of light can exist between them.

Definition 3.1.12 If motion from one event to another does not require speeds greater than c, then the events are said to be *timelike* separated, or causally separated. For two timelike events, one is in *the future lightcone* of the other, and the former is in *the past lightcone* of the latter. The future light cone of event x^μ at time t is the union of all lightlike 4-vectors anchored at x^μ and pointing to events with time $t' > t$, and the past light cone is the union of all such 4-vectors pointing to events with time $t' < t$. Events that are not timelike or lightlike separated are *spacelike* separated, or acausally separated. Such events are said to lie *elsewhere* with respect to one another.

Remark 3.1.13 Definitions for spacelike- and timelikeness based on the sign of Δs^2 depend on the choice to assign the Lorentz group as O(1,3) or O(3,1). Such definitions appear in Section 3.3. The definition of proper time in terms of the spacetime interval also depends on this choice, so it will appear in Section 3.3 as well.

Definition 3.1.14 Proper time in a given reference frame is the time measured on clock which is stationary in that frame.

Example 3.1.15 TWIN PARADOX

Definition 3.1.16 The relativistic energy and momentum of a massive particle are

$$E = \gamma mc^2 \quad , \quad \text{and} \qquad \vec{p} = \gamma m\vec{u} \quad ,$$

where the 3-velocity $\vec{u}$ of the particle is not to be confused with the 3-velocity $\vec{v}$ between two Lorentz frames. The kinetic energy is

$$E_k = mc^2\big(\gamma - 1\big) \quad .$$

The rest energy is mc^2, which is also called *the invariant mass*, so the kinetic energy is the difference between the total relativistic energy and the rest energy.

Remark 3.1.17 On-shell and off-shell for massive and massless particles

§3.2 Riemannian Manifolds

It will not be our intention to get too deep in the jargon of the mathematical underpinnings of relativity, but a Lie group such as the Lorentz group has an associated manifold, and we want to study a representation other than the usual one. To demonstrate the well-motivation of this research line, we will show that the new manifold presented here is diffeomorphic to (the same as) the better-studied one. To present at least the veneer of rigor in this task, we will need some mathematical definitions beyond the bare minimum needed to do physics in special relativity. Most of that is condensed in this section.

Wald describes a manifold as follows [6].

> "An n-dimensional manifold is a set that has the local differential structure of $\mathbb{R}^n$ but not necessarily its global properties."

This opens the door to the connection between vectors $x^\mu \in \mathbb{R}^4$ equipped the vector product $x^\mu y_\mu = \eta_{\mu\nu} x^\mu y^\nu$ where $\eta_{\mu\nu}$ is a Lorentzian metric, and vectors $x^\mu \in \mathcal{H}$ equipped with the vector product $x^\mu y_\mu = \gamma_{\mu\nu} x^\mu y^\nu$ where $\gamma_{\mu\nu}$ is a Euclidean metric. A first separation of a general manifold is the Riemannian manifold: a set equipped with a positive-definite inner product. This does not suffice for the manifold of special relativity because we want the inner products of our 4-vectors to be spacelike, null, and timelike corresponding to positive, negative, and vanishing inner products in some assignment convention. Thus, we arrive at the pseudo-Riemannian manifold. The restriction that the inner product should be positive-definite is relaxed, and instead it is said to be non-degenerate, meaning that if $x^\mu y^\nu g_{\mu\nu} = 0$ for any y^μ, then x^μ is the zero vector. While it not presently be the case that we intend to restate the foundations of analysis (or the foundations of algebra in the next section), a nice treatment of manifolds as topological spaces may be found in Appendix A of Wald [6].

ALSO [7]

> "We write local coordinates on any open subset $U \subset M$ as $(x^1, \ldots, x^n)$, x^i, or x, depending on context. Although, formally speaking, coordinates constitute a map from U to $\mathbb{R}^n$, it is more common to use a coordinate chart to identify U with its image in $\mathbb{R}^n$, and to identify a point in U with its coordinate representation (x^i) in $\mathbb{R}^n$."

Definition 3.2.1 A manifold is...

If a manifold is equipped with a metric then we assign to the manifold a bilinear form...

Remark 3.2.2 1. What is a manifold? A manifold is a mathematical space that locally resembles Euclidean space. In other words, it is a topological space that is smooth and can be described using coordinates. 2. What is the dimensionality of a manifold? The dimensionality of a manifold is the number of coordinates needed to describe a point on the manifold. For example, a 2-dimensional manifold would require two coordinates, such as latitude and longitude on the surface of a sphere. 3. What is a metric on a manifold? A metric on a manifold is a way of measuring distances between points on the manifold. It defines the notion of distance and angle on the manifold, similar to how the Pythagorean theorem defines distances in Euclidean space. 4. What is the relationship between a manifold and a metric? A metric is an essential component of a manifold. It allows us to define distances and angles on the manifold, which in turn allows us to study the properties and geometry of the manifold. 5. How are manifolds and metrics used in physics? Manifolds and metrics are used extensively in physics, especially in the fields of relativity and quantum mechanics. They are used to describe the geometry of space and time, and to study the behavior of particles and forces within this space. They are also used in various other areas of physics, such as fluid dynamics and thermodynamics.

Reference: https://www.physicsforums.com/threads/manifold-and-metric-answers-to-your-questions.199426/

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Not every manifold is a vector space A manifold is a set of points that locally resembles a Euclidean space. A vector space is a set of objects that can be added and scaled. A vector manifold is a set of vectors that generate tangent spaces to the manifold1. A vector field is a function that assigns a tangent vector to each point of the manifold2. A vector space can be regarded as a manifold, but not every manifold is a vector space34.

`https://www.bing.com/search?q=is+every+vector+space+a+manifold`

Remark 3.2.3 proof that finite dimensional vector spaces are manifold

`https://math.stackexchange.com/questions/2723332/proof-that-finite-dimensional-v`

Definition 3.2.4 A Riemannian manifold is a manifold equipped with a positive-definite metric. Positive-definiteness means

Definition 3.2.5 A Pseudo-Riemannian manifold is a manifold equipped with a non-degenerate metric. Non-degenerate means

Remark 3.2.6 xx

> "As as a rule of thumb, proofs that depend only on the invertibility of the metric tensor, such as existence and uniqueness of the Riemannian connection and geodesics, work fine in the pseudo-Riemannian setting, while proofs that use positivity in an essential way, such as those involving distance-minimizing properties of geodesics, do not."

So, if anything more than an uncountable number of sign errors related to terms like $e^{ik^\mu \cdot x^\mu}$ and $\Box^2\psi$ is lost in going to the $\mathcal{H}$ representation from $\mathbb{R}^4$ (unlikely), we know that somethings are gained.

Theorem 3.2.7 *The Minkowski metric $\eta_{\mu\nu}$ is non-degenerate but not positive-definite.*

Remark 3.2.8 We have to get rid of the positive-definite to have topologically separated lightlike, spacelike, and timelike regions.

Theorem 3.2.9 *The Euclidean metric γ_{ij} positive definite.*

Remark 3.2.10 In the conventions of Carroll [2], any metric with all the same signs is "Euclidean," and any metric with one sign different from the others "Lorentzian." However, the flat metric with signature $\{----\}$ is not positive-definite, and the Pseudo-Riemannian manifold $(\mathbb{R}^4, \text{diag}(-1,-1,-1,-1))$ is not identically $\mathbb{R}^4$. For lack of a better word, we will call this metric *pseudo-Euclidean.*

Definition 3.2.11 The negative Euclidean metric is

Theorem 3.2.12 *It is possible to associate a vector space with the negative Euclidean metric so that vectors in the resultant manifold form a representation of the Lorentz group.*

Remark 3.2.13 POINCARE CONJECTURE: This was the foundation for much of the program in the MCM anyways.

Remark 3.2.14 Finite extinction time for the solutions to the Ricci flow on certain three-manifolds by Grisha Perelman

https://arxiv.org/pdf/math/0307245.pdf

IMPORTANT FOR $\varnothing$???

[A-G] S.Altschuler, M.Grayson Shortening space curves and flow through singularities. Jour. Diff. Geom. 35 (1992), 283-298.

§3.3 Real and Complex 4-vectors

In this section, we want to show the diffeomorphism of the pseudo-Riemannian manifolds $\{\mathbb{R}^4, \eta\}$ and $\{\mathcal{H}, \gamma\}$. Matrix groups are all manifolds, and if manifolds M and N are diffeomorphic, then the elements of M and N are said to be representations of the group.

Distinguish vector inner product from the manifold inner product. These things are the same in the $\mathbb{R}^4$ representation of the Lorentz group, but the two inner products will be sign inverted when we use the $\mathcal{H}$ representation. The inner product using the complex conjugate is natural to quantum theory but not really relativity, so we open the door toward new ways of describing spinors and also new tools applications toward quantum gravity. The main difference between vectors and spinors is that rotation by 2π, usually an identity operation, becomes a sign inversion operation: a spinor is an eigenfunction of OPERATOR with eigenvalue -1.

Definition 3.3.1 In Euclidean analysis, a vector $\vec{x} \in \mathbb{R}^n$ is written

$$\vec{x} = (x_1, x_2, \ldots, x_n) \quad ,$$

where the tuple notation abbreviates pairwise multiplication with a unit vector basis spanning $\mathbb{R}^n$:

$$(x_1, x_2, \ldots, x_n) \equiv x_1\hat{e}_1 + x_2\hat{e}_2 + \cdots + x_n\hat{e}_n \quad .$$

IN EUCLIDEAN SPACE $\mathbb{R}^n$, THE INNER PRODUCT IS THE DOT PRODUCT

Definition 3.3.2 In Euclidean vector analysis, the inner product of any 2 vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$ is a positive-definite quantity. In the Euclidean case, this is often written with the dot product:

$$\langle \vec{x}, \vec{y} \rangle \equiv \vec{x} \cdot \vec{y} = (x_1, x_2, \ldots, x_n) \cdot (y_1, y_2, \ldots, y_n) = \sum_{k=1}^{n} x_k y_k \quad .$$

The dot product is the sum of pairwise products of the entries. Since the metric is the Euclidean one, there is no concern for the placement of upper or lower indices: all indices behave like matrix indices.

Definition 3.3.3 In tensor analysis, the components of a vector are strictly written with an upper index; the entries of the n-vector $\vec{x} = x^k\hat{e}_k$ are x^k. As with the parenthesis notation, by convention we omit the basis vectors and focus on the components x^k. The inner product of $x^k, y^j \in \mathbb{R}^4$ is their contraction with the Euclidean metric γ_{ij}:

$$\langle x^k, y^j \rangle \equiv x^k y^j \gamma_{kj} = x^1k^1 + x^2k^2 + \cdots + x^ny^n = \sum_{k=1}^{n} x_k y_k \quad .$$

This is exactly equal to the dot product.

FINISH

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Show that the inner product is not a Lorentz scalar! The Lorentz scalar is the contraction with the metric. LS is all we care about.

KKKKKKKKKKKKKKKKKKKKKKk

Definition 3.3.4 The inner product of complex vectors $\vec{u}, \vec{v} \in \mathbb{C}$ is defined as

$$\big\langle \vec{u}, \vec{v} \big\rangle \equiv x^k y^j \gamma_{kj} = x^1 k^1 + x^2 k^2 + \cdots + x^n y^n \quad .$$

The inner product for $x^\mu \in \mathcal{H}$ is defined as

$$\big(x^\mu\big)^2 \equiv |x^\mu|^2 = \big(x^\mu\big)^* x_\mu = \big(x^\mu\big)^* \eta_{\mu\nu} x^\nu$$

ETA/GAMMA?????

MORE THAN ONE KIND OF INNER PRODUCT IS POSSIBLE!!! OFTEN ONE SPEAKS OF **THE** INNER PRODUCT, SO DISTINCTIONS MUST BE MADE. WE WILL CALL THE PRODUCT THAT DETERMINES THE MINKOWSKI SQUARE, “THE MINKOWSKI PRODUCT”

Remark 3.3.5 Inner product is positive definite. Since we want timelike, null, and spacelike vectors, this will not suffice. We need another product.

GENERAL BILINEAR FORM

COPY CARROLL p23

If $x^\mu \in \mathbb{R}^4$, then the action of the inner product on two vectors is the inner product. The inner product of two complex vectors $x^\mu \in \mathcal{H}$ will be positive definite when it is defined in the usual way

$$\big(x^\mu\big)^2 = \sum_\mu \big(x^\mu\big)^* x^\mu$$

Definition 3.3.6 An axial vector is KKKKKKKKKKKK

Definition 3.3.7 A polar vector is KKKKKKKKKKKKK

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FOLLOW CARROLL: Maps between manifolds. There exists a function of a complex variable that takes $\widetilde{x}^\mu \in \mathbb{R}^4$ to $x^\mu \in \mathcal{H}$.

KKKKKKKKKKKKKKKKKKKKKKKKK

Article 3.3.8 Now we will consider a complex 4-vector as

$$x^\mu = (it, x, y, z) \quad , \quad \text{or} \qquad x^\mu = (t, ix, iy, iz)$$

The Lorentz invariant scalar product still

$$\big(x^\mu\big)^2 = x^\mu x_\mu = x^\mu\big(\eta_{\mu\nu}x^\nu\big) \quad ,$$

but now the metric is the 4D Euclidean metric

$$g_{\mu\nu} = \gamma_{\mu\nu} = \text{diag}(\pm1, \pm1, \pm1, \pm1) \quad .$$

For x^μ with imaginary time part, the inner product is computed as

$$\begin{aligned} x^\mu\big(\gamma_{\mu\nu}x^\nu\big) &= (it, x, y, z)\cdot(\pm it, \pm x, \pm y, \pm z) \\ &= \mp t^2 \pm \big(x^2+y^2+z^2\big) \end{aligned}$$

There exists a null interval in the convention for complex 4-vectors with imaginary time parts. For complex 4-vectors with imaginary space parts, we have

$$\begin{aligned} x^\mu\big(\gamma_{\mu\nu}x^\nu\big) &= (t, ix, iy, iz)\cdot(\pm t, \pm ix, \pm iy, \pm iz) \\ &= \pm t^2 \mp \big(x^2+y^2+z^2\big) \end{aligned}$$

The freedom to choose the space part or the time part as imaginary reflects the freedom to choose the metric signature from $\{\pm\mp\mp\mp\}$ in the previous article. However, since we still have freedom to choose the sign convention of the Euclidean metric from $\{\pm\pm\pm\pm\}$, there should be some new physical freedom associated. We will investigate this.

§3.4 Submanifolds in the MCM Unit Cell

COPY SUBMANIFOLDS FROM CARROLL

What we have called the manifold of special relativity so far is called "one quantum of spacetime" in the MCM, and the spectra of such quanta are said to be the standard model particles [5,8]. Via the introduction of a new application for holographic duality, the 4D manifold of physical observables in spacetime, called $\mathcal{H}$, is embedded between two 5D spaces $\Sigma^\pm$, as in Figure [?]. $\mathcal{A}$ and Ω are also 4-spaces with elements that should form a representation of the Lorentz group. Most excitingly, the MCM new fifth dimension $\chi^4_\pm$ (horizontal on the page) is left-handed or right-handed with respect to the coordinates $\mathcal{H}$. We will want to form 4-vectors in $\Sigma^\pm$ by taking the 3D space part together with either the chronological or chirological time. The 2×2 matrix representation of 4-vectors will be very natural for this because given a 5-vector

$$\chi^A = (\chi^0, \chi^1, \chi^2, \chi^3, \chi^4) \quad ,$$

there exists two 2×2 matrices

$$\widehat{\chi}_0 = \begin{bmatrix} \chi^0+\chi^3 & \chi^1-i\chi^2 \\ \chi^1+i\chi^2 & \chi^0-\chi^3 \end{bmatrix} \quad , \quad \text{and} \qquad \widehat{\chi}_4 = \begin{bmatrix} \chi^4+\chi^3 & \chi^1-i\chi^2 \\ \chi^1+i\chi^2 & \chi^4-\chi^3 \end{bmatrix} \quad .$$

Both of these are rank-2 spinors (CHECK)!!! By developing the different properties in of such matrices in the $\widehat{\chi}_\pm$ variants, we will want to obtain the left- and right-handed Weyl spinors ψ_R^α and $\bar{\chi}_L^{\dot{\alpha}}$ used to construct Dirac bispinors for the observable quantum physics of relativistic charged particles in $\mathcal{H}$.

Carroll gives the example of something that is not a manifold as a line terminating on a plane, but $\chi_\pm^4$ are positive- and negative-definite, so we may treat them as manifolds.

In some way, we will want the property of spinors that rotation by 2π results in sign inversion to reflect the sign of the fifth position in the metric signature.

==============================

We have set up the MCM unit cell to be such that the fundamental SM particles are taken as quanta of spacetime: a 3-space joined to a chronological or chirological time part. We have raised some questions about the metric discontinuity between $\Sigma^\pm$ where the $A^\mu = 0$ metric $g_{AB}^\pm$ has signature $\{+---\pm\}$ or $\{-+++\pm\}$ when $\chi^A \in \mathbb{R}^5$. If evolution along $\chi_\pm^4$ is timelike in $\Sigma^\pm$, then it is spacelike in $\Sigma^\mp$, and there is no way to construct a smooth evolution. So, by moving to a manifold in which the vectors are nos strictly real-valued, we migrate to another convention where a solution is not ruled out a priori.

======================

If $x^\mu \in M$ is a position vector, it belongs to $\mathcal{H}$, but the 4-momentum $p^\mu \in M$ belongs to the tangent space to $\mathcal{H}$. Since we are dealing with flat space $\mathcal{H}$ is its own tangent space at every point, but this is not the case in general. For $\mathcal{A}$ and Ω, $x_\pm^\mu, p_\pm^\mu \in M$ are such that

$$x_+^\mu \in \Omega$$
$$p_+^\mu \in T\Omega$$
$$x_-^\mu \in \mathcal{A}$$
$$p_-^\mu \in T\mathcal{A} \quad ,$$

where the T indicates the tangent bundle, which is the union of the tangent spaces at every point in $\mathcal{A}$ or Ω.

Maybe say something about the cotangent bundle.

===========================

Define a Frenet frame with $x^0, \chi_\pm^4, x^i$????

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N},$$
$$\frac{d\mathbf{N}}{\mathrm{d}s} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$$

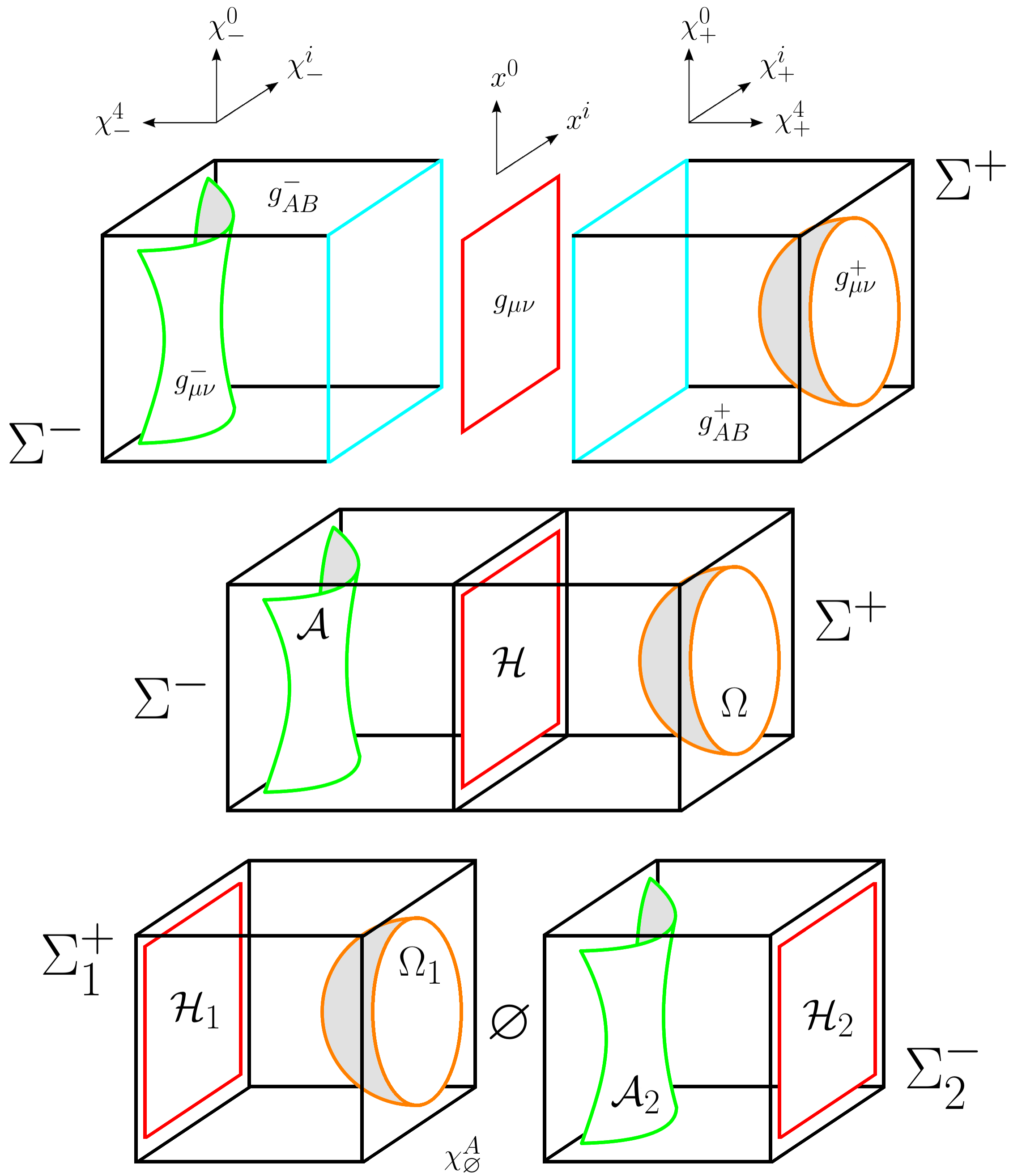

Figure 1: Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Ut purus elit, vestibulum ut, placerat ac, adipiscing vitae, felis. Curabitur dictum gravida mauris. Nam arcu libero, nonummy eget, consectetuer id, vulputate a, magna. Donec vehicula augue eu neque. Pellentesque habitant morbi tristique senectus et netus et malesuada fames ac turpis egestas. Mauris ut leo. Cras viverra metus rhoncus sem. Nulla et lectus vestibulum urna fringilla ultrices. Phasellus eu tellus sit amet tortor gravida placerat. Integer sapien est, iaculis in, pretium quis, viverra ac, nunc. Praesent eget sem vel leo ultrices bibendum. Aenean faucibus. Morbi dolor nulla, malesuada eu, pulvinar at, mollis ac, nulla. Curabitur auctor semper nulla. Donec varius orci eget risus. Duis nibh mi, congue eu, accumsan eleifend, sagittis quis, diam. Duis eget orci sit amet orci dignissim rutrum.

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}$$

§4 Lorentz Transformations

Lorentz transformations preserve objects' Minkowski length in special relativity, and the transformations form a group under matrix multiplication. Often one associates 4-vector $x^\mu \in \mathbb{R}^4$ with an event, but

$$\forall x^\mu = \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \qquad \exists \widehat{x} = \begin{bmatrix} t+z & x-iy \\ x+iy & t-z \end{bmatrix} \qquad \text{s.t.} \qquad \big(x^\mu\big)^2 = \det \widehat{x} \quad .$$

Thus, we may speak of vectors x^μ or matrices $\widehat{x}$ as representations of the Lorentz group. Linear transformations in the form

$$x^\mu \to x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu \qquad \text{s.t.} \qquad \big(x^\mu\big)^2 = \big(x^{\mu'}\big)^2$$

$$\widehat{x} \to \widehat{x}' = \Lambda \widehat{x} \qquad \text{s.t.} \qquad \det \widehat{x} = \det \widehat{x}'$$

will all be said to be Lorentz transformations. However, the elements of the Lorentz group (the Lorentz transformations) will take different forms in one representation or another. One says there is a *realization* of the group corresponding to each *representation*. Lorentz transformations on $\widehat{x}$ preserve the determinant of a 2×2 matrix, but the determinant is not defined at all for x^μ.

In this section, we will review the basic properties of the Lorentz group in the two representations shown above, and then we will develop the properties of the realization when x^μ is complex- rather than real-valued. The complex 4-vector representation will be main the object examined in this paper, so we will use the tilde to make the distinction $\widetilde{x}^\mu \in \mathbb{R}^4$ and $x^\mu \in \mathcal{H}$. The most common representation of the Lorentz group, a real-valued 4-vector, is usually called x^μ without the tilde, but here we will call that $\widetilde{x}^\mu$.

The review in this section mostly follows Jaffe [9] and Steane [10]. We will also follow the convention from Carroll [2] that primed indices represent Lorentz transformed quantities.

§4.1 Fundamentals of Lorentz Transformations

Lorentz transforms are matrices, not tensors. Although it is normal to write expressions such as

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu \quad ,$$

for the Lorentz transformation of a 4-vector, there is no tensorial significance given to the placement of the upper and lower indices. Rather, the first index μ' is the row of the Λ matrix. The summation of the repeated index μ reflects the condition that matrix multiplication of an $m \times n$ matrix from the left by an $a \times b$ matrix is only a defined operation if b and m span the number of components. In matrix notation, the tensor expression $x^\mu y_\mu = y_\mu x^\mu$, which is the scalar product of a vector and a dual vector, must be written with the dual vector first:

$$y_\mu x^\mu \quad \longrightarrow \quad \mathsf{YX} = \begin{bmatrix} y_0 & y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{bmatrix} \quad ,$$

where Y is a 1×4 matrix and X is a 4×1 matrix, and the entries in each are the corresponding components of y_μ and x^μ. Most notably, however, the entries of y_μ are not the entries of y^μ because the metric is required for raising and lowering the indices. Given the Minkowski metric in signature $\{+---\}$:

$$y^\mu = (y^0, y^1, y^2, y^3) \quad \Longrightarrow \quad y_\mu = (y^0, -y^1, -y^2, -y^3) \ \ ,$$

so plainly $y^\mu \neq y_\mu$. Thus, we cannot use convenient notation in which X, Y are one-column matrices corresponding to tensorial 4-vectors such that and $y_\mu x^\mu \to \mathsf{Y}^T\mathsf{X}$. So, since the object of this section is the abstract algebraic Lorentz group rather than physics in Minkowski space, we will use matrix notation following the conventions of Steane in [10]. In general, vectors, dual vectors, and rank-2 tensors with two lower indices can be written as matrices, but the tensor-matrix correspondence is not universal and we will work in matrix notation.

===================================

The fundamental representation of the Lorentz group is a real-valued 4-vector $x^\mu \in \mathbb{R}^4$. The realization of the Lorentz group in this representation consists of boosts and rotations, and their products. Weyl spinors also form a representation of the Lorentz group. By the two-to-one SOMETHING-ism between SO(3) and SU(2), we have two different spinor representations called $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. This comes from our ability to write points in spacetime as

$$x^\mu = (x^0, x^1, x^2, x^3) \quad \longleftrightarrow \quad \tilde{x} = \begin{bmatrix} x^0 + x^3 & x^1 - ix^2 \\ x^1 + ix^2 & x^0 - x^3 \end{bmatrix}$$

The matrix is a rank-2 spinor and we pull out rank-1 spinors via the 2-to-1 correspondence (somehow).

So, while such things are very well known [1, 4, 9, 10], presently, we will use another, less common representation of the Lorentz group $x^\mu \in \mathcal{H}$. To prove certain results and make appropriate generalizations to the MCM, we will need to find the realizations of the Lorentz group for these complex vectors and the spinors constructed from them. Luckily, Roman has worked out the case for imaginary time and real space in Appendix 3-2 to [4].

"Consider the four-dimensional Minkowski–Lorentz space of events,

$$x_1 = x, \quad x_2 = y, \quad x_3 = z, \quad x_4 = ict.$$

The points of this space form the manifold of special relativity. Let

$$x_\mu \to x'_\mu = \alpha_{\mu\nu} x_\nu \tag{4.1}$$

be a homogeneous linear mapping of the space onto itself. (The usual dummy index summation convention is adopted; summation over repeated Greek indices from 1 to 4 is understood.) The coefficients of the transformation are subject to the reality conditions

$$\begin{array}{rll} \alpha_{ik}, & i,k = 1,2,3 & \textit{real}, \\ \alpha_{i4} \text{ and } \alpha_{4i}, & i = 1,2,3 & \textit{imaginary}, \\ \alpha_{44}, & & \textit{real}. \end{array}$$

Let the coefficients be further restricted by the *orthogonality conditions*

$$\alpha_{\mu\nu}\alpha_{\mu\rho} = \delta_{\nu\rho}$$
$$\alpha_{\nu\mu}\alpha_{\rho\mu} = \delta_{\nu\rho}$$

It then follows that [*these*] transformations leave the length of a Minkowski vector invariant:

$$x_\mu x_\mu \to x'_\mu x'_\mu = x_\mu x_\mu,$$

and also

$$x'_\mu x'_\mu \to x_\mu x_\mu = x'_\mu x'_\mu.$$

It can be easily seen that the set of all transformations with this property form a group. This group is called the *homogeneous Lorentz group L*. Its unit element is given by

$$\alpha_{\mu\nu} = \delta_{\mu\nu}$$

and the inverse of [*Equation (4.1)*] reads

$$x'_\mu \to x_\mu = \alpha_{\nu\mu} x'_\nu.$$

It is easy to see that [...] there are six independent quantities $\alpha_{\mu\nu}$ (three real and three imaginary). We will consider them as *parameters* of a group element."

By making timelike part of a Minkowski vector imaginary, the boost parameters which are usually described as rotation by a complex angle acquire a requisite factor of i. Other than that, the realization of the Lorentz group is the same. Below, we will lay out the full case for making space and time alternatingly imaginary in some convention, we will use the 0-index to describe time rather than the 4-index used by Roman, and we will use tensor indices rather than the all-lower matrix indices used by Roman.

For the MCM specifically, we have many times raised a question about how timelike motion in one of $\Sigma^{\pm}$ might be linked to spacelike motion in $\Sigma^{\mp}$ at $\mathcal{H}$ or $\varnothing$ when the metric signature in $\Sigma^{\pm}$ is $\{+---\pm\}$ or $\{-+++\pm\}$. The purpose in using a complex vector representation of the Lorentz group will be to remove the non-Euclidean character of the metric (where we will call a metric Euclidean if its eigenvalues are all $+1$ or all -1). To create matching conditions for smooth evolutions across bounding branes, we will employ the alternating phase condition suggested in [5]. This will require that we alternate between the scheme for imaginary time detailed by Roman and another one in which time is real and space is imaginary. There exist many notations for inserting a conditional imaginary number into an expression, for example

$$e^{\imath\theta} = \cos\theta + i\sin\theta \equiv \text{cis}\theta$$

with an appropriate restriction on θ. However, since we will want to consider cases for 1 and i without considering -1 and $-i$, we will introduce a new symbol $\mathbb{i}^{\uparrow\downarrow}$ which allows us to write $x^\mu \in \mathcal{H}$ as

$$x^0 = \mathbb{i}^{\uparrow\downarrow}t \ \ , \ \ x^1 = \mathbb{i}^{\downarrow\uparrow}x \ \ , \ \ x^2 = \mathbb{i}^{\downarrow\uparrow}y \ \ , \ \ x^3 = \mathbb{i}^{\downarrow\uparrow}z \ \ .$$

Hopefully this cluttered notation is not too cluttered: $\mathbb{i}^{\uparrow} = 1$ and $\mathbb{i}^{\downarrow} = 1$ with $\uparrow\downarrow$ functioning as the $\pm$ symbol does. After we show the realization of the Lorentz group and prove that the manifold of special relativity works as usual in the $\uparrow$ and $\downarrow$ permutations, we will make extensions to simultaneous, different combinations in different submanifolds of the MCM unit cell, and then we introduce non-trivial new behaviors by considering the cases for

$$\mathbb{i}^{\uparrow\downarrow} \quad \to \quad \mathbb{q}^{\uparrow\downarrow} \ \ ,$$

where $\mathbb{q}^{\uparrow\downarrow}$ is used to assign real and quaternion phase rather than real and imaginary. By doing so, we will intend to construct the algebra of quantum mechanical spin operators from the phase convention on spacetime structure in the MCM unit cell. Namely, where he have defined physics in $\mathcal{H}$ as a sum of contributions from physics in $\Sigma^{\pm}$, we might examine the free particle Hamiltonian $H = xp$ in each of $\Sigma^{\pm}$ to pick up one quaternion phase or another so that the sum

$$H_{\mathcal{H}} = \frac{1}{2}\big(H_{\mathcal{A}} + H_{\Omega}\big) \ \ ,$$

begins to look like the quaternion commutator

$$[\mathbf{i}, \mathbf{j}] = \varepsilon_{ijk}\mathbf{k} \ \ .$$

Another thing we will look at is

$$H_{\mathcal{H}} = \Phi H_{\mathcal{A}} - \varphi H_{\Omega} \ \ ,$$

so that there is a remainder.

Also, since we have determined that the MCM fifth dimension $\chi^4_\pm$ is not a local variable, meaning it information and correlations are not restricted by the speed of light, we will need to consider things like boosts with boost parameters greater than one. For this reason, we will not follow the usual convention to set $c = 1$.

==

Definition 4.1.1 The flat Lorentzian metric is a rank $(0, 2)$ tensor $\eta_{\mu\nu}$ with matrix representation

$$\boldsymbol{\eta}_\pm = \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{bmatrix}$$

Sometimes in the literature the symbol for the metric stripped of its indices is taken to mean the determinant of its matrix representation, the expression $\sqrt{-g}$ for a metric $g_{\mu\nu}$ often means the square root of the sign-inverted determinant, but we will not use that convention here. Instead, we will drop the indices and use the bold typeface to indicate that the metric is being treated as a matrix, and we will always use det to indicate a determinant. Since the Minkowski metric is its own inverse, $\boldsymbol{\eta}_\pm$ is the matrix representation of $\eta^\pm_{\mu\nu}$ and $\eta^{\mu\nu}_\pm$: $\boldsymbol{\eta} = \boldsymbol{\eta}^{-1}$. From the definition of the inverse, the Minkowski metric written with one upper and one lower index is the 4×4 identity matrix:

$$\eta^\nu_\mu = \eta_{\mu\sigma}\eta^{\nu\sigma} = \delta^\nu_\mu \quad \longrightarrow \quad \boldsymbol{\eta}_\pm\boldsymbol{\eta}^{-1}_\pm = \boldsymbol{\eta}^{-1}_\pm\boldsymbol{\eta}_\pm = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \equiv \mathbb{1}_4 = \delta_{\mu\nu} \ .$$

When the sign convention is irrelevant, we will sometimes write $\boldsymbol{\eta}_\pm$ simply as $\boldsymbol{\eta}$. In one sign convention or the other, this metric is frequently referred to the in the literature as the Minkowski metric. With foresight toward comparisons with the (identically flat) Euclidean metric $\gamma_{\mu\nu} = \delta_{\mu\nu}$ (with matrix representation $\boldsymbol{\gamma} = \pm\mathbb{1}_4$) that has ±1 for all of its entries, we will refer to $\boldsymbol{\eta}$ as Lorentzian to call attention to one diagonal entry having opposite sign to the others.

Definition 4.1.2 In tensor notation, the Minkowski square of a real-valued 4-vector in flat spacetime

$$x^\mu = (t, x, y, z) \quad ,$$

is its double contraction with the metric:

$$\begin{aligned} x^2_M &= \eta_{\mu\nu}x^\mu x^\nu \\ &= (\pm t, \mp x, \mp y, \mp z) \cdot (t, x, y, z) \end{aligned}$$

$$= \pm \left(t^2 - x^2 - y^2 - z^2\right) \quad .$$

In matrix notation, we have

$$\begin{aligned}
\mathsf{X}_M^2 &= \mathsf{X}^T \boldsymbol{\eta} \mathsf{X} \\
&= \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \mp 1 & 0 & 0 \\ 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & \mp 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} \\
&= \begin{bmatrix} t & x & y & z \end{bmatrix} \begin{bmatrix} \pm t \\ \mp x \\ \mp y \\ \mp z \end{bmatrix} \\
&= \pm \left(t^2 - x^2 - y^2 - z^2\right) \quad .
\end{aligned}$$

Observe that $x_M^2 = \mathsf{X}_M^2$.[2]

Definition 4.1.3 A Lorentz transformation realized in the $\widetilde{x}^\mu$ representation is labeled $\widetilde{\Lambda}$. The Lorentz transformation of $\widetilde{x}^\mu \in \mathbb{R}^4$ is written in matrix notation as

$$\mathsf{X}' = \widetilde{\Lambda} \mathsf{X} \quad .$$

Lorentz transformations are defined to preserve Minkowski length:

$$\mathsf{X}_M^2 = \left(\mathsf{X}'\right)_M^2 \quad .$$

The invariance of the Minkowski length under Lorentz transformations is the defining property of such transformations. Given two matrix Lorentz transformations[3]

$$\begin{aligned}
\mathsf{X} \to \mathsf{X}' \quad &= \widetilde{\Lambda} \mathsf{X} \quad = \sum_k \widetilde{\Lambda}_{jk} \mathsf{X}_k \\
\mathsf{X}^T \to \left(\mathsf{X}'\right)^T &= \mathsf{X}^T \widetilde{\Lambda}^T = \sum_k \left(\mathsf{X}^T\right)_k \left(\widetilde{\Lambda}^T\right)_{jk} = \sum_k \left(\mathsf{X}^T\right)_k \left(\widetilde{\Lambda}\right)_{kj} \quad ,
\end{aligned}$$

the Minkowski square of vector X is equal to the Minkowski square of the Lorentz transformed vector X':

$$\mathsf{X}^T \boldsymbol{\eta} \mathsf{X} = \left(\mathsf{X}'\right)^T \boldsymbol{\eta} \mathsf{X}' = \left(\widetilde{\Lambda} \mathsf{X}\right)^T \boldsymbol{\eta} \left(\widetilde{\Lambda} \mathsf{X}\right) \quad .$$

Every matrix that satisfies this constraint on transformations is said to be a Lorentz transformation $\widetilde{\Lambda}$ in the $\widetilde{x}^\mu$ representation. The set of all such matrices

[2] Via these correspondences, the realization of the Lorentz group in the $\widetilde{x}^\mu$ and X representations are identical.

[3] Note that the usual convention for upper and lower indices on x^μ and x_μ is reflected in the alternating convention for matrix multiplication from the left and right when Lorentz transforming X and X^T.

is labeled $\widetilde{\mathcal{L}}$, and $\widetilde{\mathcal{L}}$ together with the matrix multiplication operation is a representation of the Lorentz group. In the language of group theory, this group is called O(3,1) or O(1,3) depending on the choice of metric signature: $\{+---\}$ or $\{-+++\}$.

Theorem 4.1.4 *For any $\widetilde{\Lambda} \in \widetilde{\mathcal{L}}$, the flat Lorentzian metric $\boldsymbol{\eta}$ satisfies*

$$\boldsymbol{\eta} = \widetilde{\Lambda}^T \boldsymbol{\eta} \widetilde{\Lambda} \quad .$$

Proof. Per Definition 4.1.3, the invariance of X_M^2 requires

$$\mathsf{X}_M^2 = \big(\mathsf{X}'\big)_M^2 \qquad \Longleftrightarrow \qquad \mathsf{X}^T \boldsymbol{\eta} \mathsf{X} = \big(\widetilde{\Lambda}\mathsf{X}\big)^T \boldsymbol{\eta} \big(\widetilde{\Lambda}\mathsf{X}\big) \quad .$$

The transpose of a product is the product of the transposes in reverse order, so we may write

$$\mathsf{X}^T \boldsymbol{\eta} \mathsf{X} = \big(\mathsf{X}^T \widetilde{\Lambda}^T\big) \boldsymbol{\eta} \big(\widetilde{\Lambda}\mathsf{X}\big) \quad .$$

Matrix multiplication operations may be carried out in any order, so introduce $\boldsymbol{\eta}' = \Lambda^T \boldsymbol{\eta} \widetilde{\Lambda}$ to write

$$\mathsf{X}^T \boldsymbol{\eta} \mathsf{X} = \mathsf{X}^T \big(\widetilde{\Lambda}^T \boldsymbol{\eta} \widetilde{\Lambda}\big) \mathsf{X} = \mathsf{X}^T \boldsymbol{\eta}' \mathsf{X} \quad .$$

This equation is only satisfied if $\boldsymbol{\eta}$ is equal to $\boldsymbol{\eta}'$. Since $\boldsymbol{\eta}' = \Lambda^T \boldsymbol{\eta} \widetilde{\Lambda}$, the theorem is proven.

Corollary 4.1.5 *For any $\widetilde{\Lambda} \in \widetilde{\mathcal{L}}$, any Lorentzian metric $g_{\mu\nu}$ (not necessarily the flat metric $\eta_{\mu\nu}$) with matrix representation $\boldsymbol{g}$ satisfies*

$$\boldsymbol{g} = \widetilde{\Lambda}^T \boldsymbol{g} \widetilde{\Lambda} \quad .$$

Proof. Definition 4.1.2 gives the Minkowski square in flat spacetime, but Lorentz transformations $\widetilde{\Lambda} \in \widetilde{\mathcal{L}}$ preserve the Minkowski square of $\widetilde{x}^\mu$ under all physical curvature conditions given by the arbitrary Lorentzian metric $g_{\mu\nu}$ (with matrix representation $\boldsymbol{g}$.) This is stated as

$$x_M^2 = g_{\mu\nu} x^\mu x^\nu \quad , \qquad \text{or} \qquad \mathsf{X}_M^2 = \mathsf{X}^T \boldsymbol{g} \mathsf{X} \quad .$$

Following Theorem 4.1.4, the matrix expression yields $\boldsymbol{g} = \widetilde{\Lambda}^T \boldsymbol{g} \widetilde{\Lambda}$ directly. This proves the corollary.

Theorem 4.1.6 *LT has real-valued entries*

Theorem 4.1.7 *LT has six params*

Theorem 4.1.8 *The inverse is also in $\widetilde{\mathcal{L}}$*

Definition 4.1.9 A Lorentz invariant is...

Contrast scalars in general with the components of vectors being scalars. Scalars are not also Lorentz invariant!

Example 4.1.10 TIME DILATION AGAIN!!!

Every object is at rest in its own reference frame. When $\vec{v} = 0$, the 4-velocity is

$$U^\mu = (U^0, 0, 0, 0) \quad .$$

§4.2 The Real Vector Representation

maybe add $x^\mu \in \mathbb{R}^4$ to title

Theorem 4.2.1 *The inner product of $x^\mu \in \mathbb{R}^4$ with itself is a Lorentz scalar.*

Proof. For a real 4-vector x^μ, the rotation matrix is

$$\Lambda^{\mu'}_{\ \nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so

$$x^{\mu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{C} & \mathrm{S} & 0 \\ 0 & -\mathrm{S} & \mathrm{C} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ x\mathrm{C} + y\mathrm{S} \\ y\mathrm{C} - x\mathrm{S} \\ z \end{bmatrix} \quad .$$

We observe that the inner product is Lorentz invariant:

$$\begin{aligned} \left(x^{\mu'}\right)^2 &= x^{\mu'} \eta_{\mu\nu} x^{\nu'} \\ &= \begin{bmatrix} t \\ x\mathrm{C} + y\mathrm{S} \\ y\mathrm{C} - x\mathrm{S} \\ z \end{bmatrix}^T \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{bmatrix} \begin{bmatrix} t \\ x\mathrm{C} + y\mathrm{S} \\ y\mathrm{C} - x\mathrm{S} \\ z \end{bmatrix} \\ &= \begin{bmatrix} t \\ x\mathrm{C} + y\mathrm{S} \\ y\mathrm{C} - x\mathrm{S} \\ z \end{bmatrix}^T \begin{bmatrix} \pm t \\ \mp\left(x\mathrm{C} + y\mathrm{S}\right) \\ \mp\left(y\mathrm{C} - x\mathrm{S}\right) \\ \mp z \end{bmatrix} \\ &= \pm t^2 \mp \left[\left(x\mathrm{C} + y\mathrm{S}\right)^2 + \left(y\mathrm{C} - x\mathrm{S}\right)^2 + z^2\right] \\ &= \pm t^2 \mp \left[\left(x^2\mathrm{C}^2 + 2xy\mathrm{CS} + y^2\mathrm{S}^2\right) + \left(y^2\mathrm{C}^2 - 2xy\mathrm{CS} + x^2\mathrm{S}^2\right) + z^2\right] \end{aligned}$$

$$= \pm t^2 \mp \left[x^2\big(\mathrm{C}^2+\mathrm{S}^2\big)+y^2\big(\mathrm{S}^2+\mathrm{C}^2\big)+z^2\right]$$
$$= \pm t^2 \mp \big(x^2+y^2+z^2\big)$$

It is properly invariant under rotations.

§4.3 The Spinor Representation

maybe add $\widehat{x} \in \boldsymbol{H}^2$ to title (check symbol)

§4.4 The Complex Vector Representation

maybe add $\widetilde{x}^\mu \in \mathcal{H}$ to title

Group theory, c.f.: [4], Appendix 2.

Review JAFFE, maybe add stuff from STEANE: [9, 10]

Definition 4.4.1 The symbol $\mathbb{i}^{\uparrow\downarrow}$ is such that

$$\mathbb{i}^{\uparrow} = i \quad , \quad \text{and} \qquad \mathbb{i}^{\downarrow} = 1 \quad .$$

The superscript $\uparrow\downarrow$ functions like the $\pm$ symbol, so

$$\mathbb{i}^{\uparrow\downarrow}\mathbb{i}^{\downarrow\uparrow} = i \quad .$$

The complex conjugation behavior is

$$\mathbb{i}^{\uparrow *} = -i \quad , \quad \mathbb{i}^{\downarrow *} = 1 \qquad \Longrightarrow \qquad |\mathbb{i}^{\uparrow\downarrow}|^2 = \big(\mathbb{i}^{\uparrow\downarrow}\big)^*\mathbb{i}^{\uparrow\downarrow} = 1 \quad .$$

The direct (non-conjugated) square of this symbol is

$$\big(\mathbb{i}^{\uparrow}\big)^2 = -1 \quad , \quad \text{and} \qquad \big(\mathbb{i}^{\downarrow}\big)^2 = 1 \quad .$$

This symbol will allow us to consider the cases for imaginary time and space parts of 4-vectors in unified expressions (unlike the previous article). Since we will want to consider sign and phase permutations separately, the more obvious notation $\mathbb{i}^{\pm}$ will not suffice. While it might be more convenient to use 10 and 01 than $\uparrow\downarrow$ and $\downarrow\uparrow$, we will want to link these cases to "spin up" and "spin down" later, and the notation is introduced with foresight. The reader is so advised.

Definition 4.4.2 To reverse the order of the arrows on the $\mathbb{i}^{\uparrow\downarrow}$ symbol, we will have to introduce one more symbol $\beta^{\uparrow\downarrow}$ such that

$$\mathbb{i}^{\uparrow\downarrow}\beta^{\downarrow\uparrow} = \mathbb{i}^{\downarrow\uparrow} \qquad \Longrightarrow \qquad \begin{cases} \beta^{\uparrow} = i \\ \beta^{\downarrow} = -i \quad . \end{cases}$$

Again, it will not suffice to use $\mp i$ because the $\pm$ symbol will be varied separately from the $\uparrow\downarrow$ symbol, and we have chosen the convention to be such that "beta down" is negative while "beta up" is positive. It follows that

$$\mathbb{i}^{\uparrow\downarrow}\beta^{\uparrow\downarrow} = -\mathbb{i}^{\downarrow\uparrow} \qquad \Longrightarrow \qquad \pm\beta^{\uparrow\downarrow} = \mp\beta^{\downarrow\uparrow} \quad .$$

MAKE LIST OF OTHER PROPERTIES
ALSO MAKE TABLE

Definition 4.4.3 The complex vector representation of points $x^\mu \in \mathcal{H}$ in Minkowski space is

$$
\begin{aligned}
x^0 &= \mathbb{i}^{\upharpoonleft\!\downharpoonright} ct \\
x^1 &= \mathbb{i}^{\downharpoonleft\!\upharpoonright} x \\
x^2 &= \mathbb{i}^{\downharpoonleft\!\upharpoonright} y \\
x^3 &= \mathbb{i}^{\downharpoonleft\!\upharpoonright} z \quad .
\end{aligned}
$$

where $t, x, y, z \in \mathbb{R}$. MAKE SOMETHING ABOUT $\widetilde{x}^\mu$

Definition 4.4.4 The metric that defines distance between complex vector points in Minkowski space is Euclidean:

$$
\gamma_{\mu\nu} = \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{bmatrix} = \pm\delta_{\mu\nu}
$$

Theorem 4.4.5 *Given the Euclidean 4-metric $\gamma_{\mu\nu}$ and a complex vector $x^\mu \in \mathcal{H}$, the line element in Minkowski space is the usual one.*

Proof. Given

$$
ds^2 = g_{\mu\nu} dx^\mu dx^\nu \quad ,
$$

we have

$$
\begin{aligned}
ds^2 &= \gamma_{\mu\nu} dx^\mu dx^\nu = \pm\delta_{\mu\nu} \\
&= \pm\left[\left(\mathbb{i}^{\upharpoonleft\!\downharpoonright} cdt\right)^2 + \left(\mathbb{i}^{\downharpoonleft\!\upharpoonright} dx\right)^2 + \left(\mathbb{i}^{\downharpoonleft\!\upharpoonright} dy\right)^2 + \left(\mathbb{i}^{\downharpoonleft\!\upharpoonright} dz\right)^2 \right] \quad .
\end{aligned}
$$

The cases of $\uparrow\downarrow$ are

$$
\begin{aligned}
ds^2_\uparrow &= \pm\left[\left(\mathbb{i}^{\uparrow} cdt\right)^2 + \left(\mathbb{i}^{\downarrow} dx\right)^2 + \left(\mathbb{i}^{\downarrow} dy\right)^2 + \left(\mathbb{i}^{\downarrow} dz\right)^2 \right] \\
&= \pm\left[i^2 c^2 dt^2 + dx^2 + dy^2 + dz^2 \right] \\
&= \pm\left[-c^2 dt^2 + dx^2 + dy^2 + dz^2 \right] \quad ,
\end{aligned}
$$

and

$$
\begin{aligned}
ds^2_\downarrow &= \pm\left[\left(\mathbb{i}^{\downarrow} cdt\right)^2 + \left(\mathbb{i}^{\uparrow} dx\right)^2 + \left(\mathbb{i}^{\uparrow} dy\right)^2 + \left(\mathbb{i}^{\uparrow} dz\right)^2 \right] \\
&= \pm\left[c^2 dt^2 + i^2 dx^2 + i^2 dy^2 + i^2 dz^2 \right]
\end{aligned}
$$

$$= \pm \left[c^2dt^2 - dx^2 - dy^2 - dz^2\right] \quad .$$

These expressions are identical they agree with the line element obtained from the Lorentzian metric $\eta_{\mu\nu} = \text{diag}(\pm 1, \mp 1, \mp 1, \mp 1)$ when points in Minkowski space are specified with $x^\mu \in \mathbb{R}^4$:

$$ds^2 = -c^2dt^2 + \sum_i \left(dx^i\right)^2 \quad , \quad \text{or} \qquad ds^2 = c^2dt^2 - \sum_i \left(dx^i\right)^2 \quad .$$

Definition 4.4.6 The Minkowski square of $\widetilde{x}^\mu$ is the scalar product

$$\begin{aligned}
\widetilde{x}^2_M &= \gamma_{\mu\nu}\widetilde{x}^\mu\widetilde{x}^\nu \\
&= \pm\left[\left(\mathbb{i}^{\updownarrows}\right)^2 t^2 + \left(\mathbb{i}^{\downuparrows}\right)^2 x^2 + \left(\mathbb{i}^{\downuparrows}\right)^2 y^2 + \left(\mathbb{i}^{\downuparrows}\right)^2 z^2\right] \\
&= \left(\mathbb{i}^{\downuparrows}\right)^2 t^2 + \left(\mathbb{i}^{\updownarrows}\right)^2 \left[x^2 + y^2 + z^2\right]
\end{aligned}$$

We will need to find the set of Lorentz transformations that leaves this invariant. Then we will call this quantity "a Lorentz scalar." The positive, negative, and vanishing conditions that determine whether $\widetilde{x}^\mu$ is timelike, spacelike, or null, will depend on the chosen conventions. The important thing is that $(\mathbb{i}^{\updownarrows})^2$ and $(\mathbb{i}^{\downuparrows})^2$ are oppositely signed, so the required behavior is available in some form.

Remark 4.4.7 NOTE THAT THIS IS NOT THE INNER PRODUCT!!!

Definition 4.4.8 The elements of the realization of the Lorentz group in the complex-valued 4-vector representation are $\widetilde{\Lambda}^{\mu'}_{\mu}$:

$$\begin{aligned}
\widetilde{\Lambda}^0_0 &= \Lambda^0_0 \\
\widetilde{\Lambda}^0_i &= \beta^{\updownarrows}\Lambda^0_i \\
\widetilde{\Lambda}^i_0 &= \beta^{\downuparrows}\Lambda^i_0 \\
\widetilde{\Lambda}^i_j &= \Lambda^i_j \quad .
\end{aligned}$$

Note that these agree with the conventions given in 4.1. The freedom to choose real or imaginary space or time in $\widetilde{x}$ is reflected in the choice of sign for the imaginary parts of $\widetilde{\Lambda}$. Recall that the Lorentz transformation is not a tensor. It is a transformation and we don't need to worry about raising and lowering operations changing the signs of the entries. These are matrices with an upper index indicating a row, and a lower index indicating a column.

Remark 4.4.9 Every element of the Lorentz group has an inverse

$$x^{\mu'} = \Lambda^{\mu'}_{\mu}x^\mu \qquad \Longrightarrow \qquad x^\nu = \left(\Lambda^{-1}\right)^\nu_{\mu'}\Lambda^{\mu'}_{\mu}x^\mu = x^\mu \quad ,$$

and every inverse Lorentz transformation belongs to the Lorentz group as well. Now we will demonstrate that realization of the Lorentz group in the complex 4-vector representation with a few brief examples.

THE METRIC IS INVARIANT

BOOSTS

ROTATIONS

SHOW THAT THE METRIC IS ITS OWN INVERSE IN THE NEW CONVENTION

Λ IS UNITARY, CHECK FOR $\widetilde{\Lambda}$

Remark 4.4.10 Since the complex 4-vector representation makes either the time or space part of a real-valued 4-vector imaginary, we should expect that $\widetilde{\Lambda}=\Lambda$ when the transformation is a pure rotation. For a boost however, which is considered to be a rotation by an imaginary angle, or a rotation of time space and time axes, then we should find $\widetilde{\Lambda}\neq\Lambda$.

Remark 4.4.11 It must be demonstrated that the complex 4-vectors transform appropriately under Lorentz transformations $x^\mu \to x'^\mu$:

$$x'^\mu = \Lambda^\mu_\nu x^\nu \quad .$$

We should also show that the Euclidean metric is invariant under Lorentz transformations:

$$\gamma'_{\mu\nu} = \Lambda^\rho_\mu \gamma_{\rho\sigma} \Lambda^\sigma_\nu = \gamma_{\mu\nu}$$

Remark 4.4.12 We will consider rotations about the z axis (in the xy-plane).

Example 4.4.13 *Spatial rotations of real and complex 4-vectors.*

In the $x^\mu \in \mathbb{R}^4$ representation, a rotation by angle θ in the xy-plane is written

$$\Lambda^{\mu'}{}_{\mu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad .$$

It follows that

$$x^{\mu'} = \Lambda^{\mu'}{}_{\mu} x^\mu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{C} & \mathrm{S} & 0 \\ 0 & -\mathrm{S} & \mathrm{C} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t \\ x\mathrm{C} + y\mathrm{S} \\ -x\mathrm{S} + y\mathrm{C} \\ z \end{bmatrix} \quad .$$

Likewise for $\widetilde{x}^\mu$:

$$\widetilde{x}^{\mu'}=\widetilde{\Lambda}^{\mu'}{}_{\mu}\widetilde{x}^\mu=\begin{bmatrix}1&0&0&0\\0&\mathrm{C}&\mathrm{S}&0\\0&-\mathrm{S}&\mathrm{C}&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\mathbb{i}^{\Uparrow\!\Downarrow}t\\\mathbb{i}^{\Downarrow\!\Uparrow}x\\\mathbb{i}^{\Downarrow\!\Uparrow}y\\\mathbb{i}^{\Downarrow\!\Uparrow}z\end{bmatrix}=\begin{bmatrix}\mathbb{i}^{\Uparrow\!\Downarrow}t\\\mathbb{i}^{\Downarrow\!\Uparrow}\big(x\mathrm{C}+y\mathrm{S}\big)\\\mathbb{i}^{\Downarrow\!\Uparrow}\big(-x\mathrm{S}+y\mathrm{C}\big)\\\mathbb{i}^{\Downarrow\!\Uparrow}z\end{bmatrix}\quad.$$

This example has confirmed the convention in Definition 4.4.8: NEED TO FIX

Example 4.4.14 *Boosts of real and complex 4-vectors.*

In the $x^\mu\in\mathbb{R}^4$ representation, a boost with rapidity ϕ in the z-direction is written

$$\Lambda^{\mu'}{}_{\mu}=\begin{bmatrix}\cosh\phi&0&0&-\sinh\phi\\0&1&0&0\\0&0&1&0\\-\sinh\phi&0&0&\cosh\phi\end{bmatrix}\quad.$$

It follows that

$$x^{\mu'}=\Lambda^{\mu'}{}_{\mu}x^\mu=\begin{bmatrix}\mathrm{Ch}&0&0&-\mathrm{Sh}\\0&1&0&0\\0&0&1&0\\-\mathrm{Sh}&0&0&\mathrm{Ch}\end{bmatrix}\begin{bmatrix}t\\x\\y\\z\end{bmatrix}=\begin{bmatrix}t\mathrm{Ch}-z\mathrm{Sh}\\x\\y\\-t\mathrm{Sh}+z\mathrm{Ch}\end{bmatrix}\quad.$$

The usual special relativity formulae are obtained by defining the boost parameter, c.f.: Carroll [2] or Jaffe [9], to be such that

$$\phi=\tanh^{-1}v\quad,$$

where v is the velocity of the boosted frame. Then

$$\cosh(\tanh^{-1}v)=\frac{1}{\sqrt{1-v^2}}\quad,\qquad\text{and}\qquad\sinh(\tanh^{-1}v)=\frac{v}{\sqrt{1-v^2}}\quad,$$

to obtain

$$\begin{aligned}t'&=\frac{t}{\sqrt{1-v^2}}-\frac{zv}{\sqrt{1-v^2}}=\gamma\big(t-zv\big)\\z'&=-\frac{tv}{\sqrt{1-v^2}}+\frac{z}{\sqrt{1-v^2}}=\gamma\big(z-vt\big)\quad,\end{aligned}$$

where

$$\gamma=\frac{1}{\sqrt{1-v^2}}\quad.$$

Obviously, the correct form of the transformation of $\widetilde{x}^\mu$ under a Lorentz boost will be

$$\widetilde{x}^\mu \to \widetilde{x}^{\mu'} = \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow}\big(t\mathrm{Ch} - z\mathrm{Sh}\big) \\ \mathbb{i}^{\downarrow\uparrow}x \\ \mathbb{i}^{\downarrow\uparrow}y \\ \mathbb{i}^{\downarrow\uparrow}\big(-t\mathrm{Sh} + z\mathrm{Ch}\big) \end{bmatrix} \quad .$$

It is trivial to determine that the requisite form of $\widetilde{\Lambda}$ is

$$\widetilde{x}^{\mu'} = \widetilde{\Lambda}^{\mu'}{}_{\mu}\widetilde{x}^\mu = \begin{bmatrix} \mathrm{Ch} & 0 & 0 & -\beta^{\uparrow\downarrow}\mathrm{Sh} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta^{\downarrow\uparrow}\mathrm{Sh} & 0 & 0 & \mathrm{Ch} \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow}t \\ \mathbb{i}^{\downarrow\uparrow}x \\ \mathbb{i}^{\downarrow\uparrow}y \\ \mathbb{i}^{\downarrow\uparrow}z \end{bmatrix} = \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow}\big(t\mathrm{Ch} - z\mathrm{Sh}\big) \\ \mathbb{i}^{\downarrow\uparrow}x \\ \mathbb{i}^{\downarrow\uparrow}y \\ \mathbb{i}^{\downarrow\uparrow}\big(-t\mathrm{Sh} + z\mathrm{Ch}\big) \end{bmatrix} \quad .$$

This example has confirmed the convention in Definition 4.4.8: NEED TO FIX

Example 4.4.15 Given a boost in the x-direction

$$\Lambda^{\mu'}_{\mu} = \begin{bmatrix} \mathrm{Ch} & \mathrm{Sh} & 0 & 0 \\ \mathrm{Sh} & \mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

the inverse is easily verified to be

$$\big(\Lambda^{-1}\big)^{\nu}_{\mu'} = \begin{bmatrix} \mathrm{Ch} & -\mathrm{Sh} & 0 & 0 \\ -\mathrm{Sh} & \mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad ,$$

Applying the conventions of Definition 4.4.8 to obtain this transformation and its inverse for $x^\mu \in \mathcal{H}$, we obtain

$$\begin{aligned} \big(\Lambda^{-1}\big)^{\nu}_{\mu'}\Lambda^{\mu'}_{\mu} &= \begin{bmatrix} \mathbb{i}^{\downarrow\uparrow}\mathrm{Ch} & -\mathbb{i}^{\uparrow\downarrow}\mathrm{Sh} & 0 & 0 \\ -\mathbb{i}^{\uparrow\downarrow}\mathrm{Sh} & \mathbb{i}^{\downarrow\uparrow}\mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\downarrow\uparrow}\mathrm{Ch} & \mathbb{i}^{\uparrow\downarrow}\mathrm{Sh} & 0 & 0 \\ \mathbb{i}^{\uparrow\downarrow}\mathrm{Sh} & \mathbb{i}^{\downarrow\uparrow}\mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} \big(\mathbb{i}^{\downarrow\uparrow}\big)^2\mathrm{Ch}^2 - \big(\mathbb{i}^{\uparrow\downarrow}\big)^2\mathrm{Sh}^2 & i\mathrm{ChSh} - i\mathrm{ShCh} & 0 & 0 \\ -i\mathrm{ShCh} + i\mathrm{ChSh} & -\big(\mathbb{i}^{\uparrow\downarrow}\big)^2\mathrm{Sh}^2 + \big(\mathbb{i}^{\downarrow\uparrow}\big)^2\mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

$$= \begin{bmatrix} \left(\mathbb{i}^{\downarrow\uparrow}\right)^2\mathrm{Ch}^2 - \left(\mathbb{i}^{\uparrow\downarrow}\right)^2\mathrm{Sh}^2 & 0 & 0 & 0 \\ 0 & -\left(\mathbb{i}^{\uparrow\downarrow}\right)^2\mathrm{Sh}^2 + \left(\mathbb{i}^{\downarrow\uparrow}\right)^2\mathrm{Ch} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Remark 4.4.16 In the real-valued $x^\mu \in \mathbb{R}^{\ntrianglerighteq}$ representation of the Lorentz group, the realization of Λ is such that all of its entries are real, and the metric satisfies

$$\Lambda^T g \Lambda = g \quad .$$

Since the realization for $\widetilde{x}^\mu \in \mathcal{H}$ contains complex numbers, we should expect that the metric is invariant between a transform and its conjugate transpose.

Theorem 4.4.17 *The Euclidean metric $\gamma_{\mu\nu} = diag\{\pm1, \pm1, \pm1, \pm1\}$ satisfies*

$$\widetilde{\Lambda}^\dagger \gamma \widetilde{\Lambda} = \gamma \quad ,$$

where $\dagger$ denotes the conjugate transpose.

Proof. Given the known behavior of the Lorentzian metric $\eta_{\mu\nu} = \mathrm{diag}\{\mp1, \pm1, \pm1, \pm1\}$

$$\eta_{\mu\nu} \to \eta_{\mu'\nu'} = \Lambda^\mu_{\mu'} \eta_{\mu\nu} \Lambda^\nu_{\nu'}$$

we may write in matrix index notation

$$\eta = \Lambda^T \eta \Lambda = \Lambda^T_{\rho\mu} \eta_{\mu\nu} \Lambda_{\nu\sigma} = \Lambda_{\mu\rho} \eta_{\mu\nu} \Lambda_{\nu\sigma} \quad .$$

The metric contains off-diagonal zeros so the constraints on the entries in Λ must come from the case in which $\mu = \nu$:

$$\eta_{\rho\sigma} = \mp\Lambda_{0\rho}\Lambda_{0\sigma} \pm \Lambda_{1\rho}\Lambda_{1\sigma} \pm \Lambda_{2\rho}\Lambda_{2\sigma} \pm \Lambda_{3\rho}\Lambda_{3\sigma}$$

Choosing the 00 component of $\eta_{\rho\sigma}$, we find

$$\mp 1 = \mp\Lambda_{00}\Lambda_{00} \pm \Lambda_{10}\Lambda_{10} \pm \Lambda_{20}\Lambda_{20} \pm \Lambda_{30}\Lambda_{30} \quad .$$

This does not agree with the orthogonality condition given by Roman in Equation (4.1):

$$\widetilde{\Lambda}_{\mu\nu}\widetilde{\Lambda}_{\mu\rho} = \widetilde{\Lambda}_{\nu\mu}\widetilde{\Lambda}_{\rho\mu} = \delta_{\nu\rho} \quad .$$

So, we can see that the $\widetilde{\Lambda}$ realization of the Lorentz group has constraints not present for Λ. Using

$$\widetilde{\Lambda}^0_0 = \Lambda^0_0 \quad , \quad \widetilde{\Lambda}^0_i = \beta^{\uparrow\downarrow}\Lambda^0_i \quad , \quad \widetilde{\Lambda}^i_0 = \beta^{\downarrow\uparrow}\Lambda^i_0 \quad , \quad \text{and} \qquad \widetilde{\Lambda}^i_j = \Lambda^i_j \quad ,$$

(as in XXXXXX) we may directly compute the theorem as

$$\gamma = \widetilde{\Lambda}^\dagger \gamma \widetilde{\Lambda}$$

$$\gamma_{\mu\nu} = \begin{bmatrix} \Lambda_{00} & \left(\beta^{\downarrow\uparrow}\right)^*\Lambda_{10} & \left(\beta^{\downarrow\uparrow}\right)^*\Lambda_{20} & \left(\beta^{\downarrow\uparrow}\right)^*\Lambda_{30} \\ \left(\beta^{\uparrow\downarrow}\right)^*\Lambda_{01} & \Lambda_{11} & \Lambda_{21} & \Lambda_{31} \\ \left(\beta^{\uparrow\downarrow}\right)^*\Lambda_{02} & \Lambda_{12} & \Lambda_{22} & \Lambda_{32} \\ \left(\beta^{\uparrow\downarrow}\right)^*\Lambda_{03} & \Lambda_{13} & \Lambda_{23} & \Lambda_{33} \end{bmatrix} \times \dots$$

$$\dots \times \begin{bmatrix} \pm\Lambda_{00} & \pm\beta^{\uparrow\downarrow}\Lambda_{01} & \pm\beta^{\uparrow\downarrow}\Lambda_{02} & \pm\beta^{\uparrow\downarrow}\Lambda_{03} \\ \pm\beta^{\downarrow\uparrow}\Lambda_{10} & \pm\Lambda_{11} & \pm\Lambda_{12} & \pm\Lambda_{13} \\ \pm\beta^{\downarrow\uparrow}\Lambda_{20} & \pm\Lambda_{21} & \pm\Lambda_{22} & \pm\Lambda_{23} \\ \pm\beta^{\downarrow\uparrow}\Lambda_{30} & \pm\Lambda_{31} & \pm\Lambda_{32} & \pm\Lambda_{33} \end{bmatrix} \quad .$$

Letting suffice again to consider only the 00 component, we obtain

$$\pm 1 = \pm\left[\Lambda_{00}\Lambda_{00} + \left(\beta^{\downarrow\uparrow}\right)^*\left(\beta^{\downarrow\uparrow}\right)\Lambda_{i0}\Lambda_{i0}\right] \quad .$$

Using $(\beta^{\downarrow\uparrow})^*(\beta^{\downarrow\uparrow}) = 1$, we obtain

$$\pm 1 = \pm\left[\Lambda_{00}\Lambda_{00} + \Lambda_{i0}\Lambda_{i0}\right] \quad .$$

Using the $\widetilde{\Lambda}_{\mu\nu}\widetilde{\Lambda}_{\mu\rho} = \delta_{\nu\rho}$ condition stated by Roman, we have

$$\pm 1 = \pm\widetilde{\Lambda}_{\mu 0}\widetilde{\Lambda}_{\mu 0} = \pm\delta_{00} \quad ,$$

which is correct. Since Lorentz transformation are *defined* to satisfy $\widetilde{\Lambda}^T g\widetilde{\Lambda} = g$, we have independently arrived at the condition cited by Roman, and theorem is proven.

Example 4.4.18 Lorentz transformations are defined to leave the Minkowski length of 4-vectors unchanged, and the property $\widetilde{\Lambda}^T g\widetilde{\Lambda}$ is sometimes taken as a consequence of that, so we will do the full calculation here as an example. Invariance of the Minkowski length under Lorentz transformations is stated as

$$\widetilde{x}^\mu\widetilde{x}_\mu = \widetilde{\Lambda}^{\mu'}_{\mu}\widetilde{x}^\mu$$

Remark 4.4.19 FROM WIKI `https://en.wikipedia.org/wiki/Lorentz_transformation`

The transformations are not defined if v is outside these limits. At the speed of light ($v = c$) γ is infinite, and faster than light ($v > c$) γ is a complex number, each of which make the transformations unphysical. The space and time coordinates are measurable quantities and numerically must be real numbers.

Remark 4.4.20 In the case of a real-valued 4-vector $x^\mu \in \mathbb{R}^4$, the inner product of x^μ with itself is a Lorentz scalar. In the case of $x^\mu \in \mathcal{H}$, the inner product, which is the dot product with the complex conjugate, or the matrix product with the conjugate transpose, the inner product is no longer a Lorentz scalar. Rather it is the contraction of $x^\mu, x^\nu \in \mathcal{H}$ with the metric which is invariant under Lorentz transformations. The key point here is that $x^\mu \in \mathbb{R}^4$

implies that the inner product is identical to contraction with the metric, but we will need to be careful to make this distinction when $x^\mu \in \mathbb{R}^4$, and we should study what will be the implications of the inner product with itself not being a Lorentz scalar.

Theorem 4.4.21 *The inner product of $x^\mu \in \mathcal{H}$ with itself is not a Lorentz scalar under rotations.*

Proof. The inner product for $x^\mu \in \mathcal{H}$ is defined as

$$\big(x^\mu\big)^2 \equiv |x^\mu|^2 = \big(x^\mu\big)^* x_\mu = \big(x^\mu\big)^* \eta_{\mu\nu} x^\nu$$

If the 4-vector is complex and the metric is Euclidean, and if we use the same rotation matrix as we have used for real 4-vectors, then

$$x^{\mu'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{C} & \mathrm{S} & 0 \\ 0 & -\mathrm{S} & \mathrm{C} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\updownarrows} t \\ \mathbb{i}^{\downuparrows} x \\ \mathbb{i}^{\downuparrows} y \\ \mathbb{i}^{\downuparrows} z \end{bmatrix} = \begin{bmatrix} \mathbb{i}^{\updownarrows} t \\ \mathbb{i}^{\downuparrows}\big(x\mathrm{C} + y\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows}\big(y\mathrm{C} - x\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows} z \end{bmatrix} \quad . \tag{4.2}$$

The inner product of the rotated complex 4-vector with itself is:

$$\begin{aligned}
\big(x^{\mu'}\big)^2 &= \big(x^{\mu'}\big)^* \gamma_{\mu\nu} x^{\nu'} \\
&= \begin{bmatrix} \mathbb{i}^{\updownarrows} t \\ \mathbb{i}^{\downuparrows}\big(x\mathrm{C} + y\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows}\big(y\mathrm{C} - x\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows} z \end{bmatrix}^\dagger \begin{bmatrix} \pm 1 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 \\ 0 & 0 & 0 & \pm 1 \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\updownarrows} t \\ \mathbb{i}^{\downuparrows}\big(x\mathrm{C} + y\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows}\big(y\mathrm{C} - x\mathrm{S}\big) \\ \mathbb{i}^{\downuparrows} z \end{bmatrix} \\
&= \begin{bmatrix} \big(\mathbb{i}^{\updownarrows}\big)^* t \\ \big(\mathbb{i}^{\downuparrows}\big)^*\big(x\mathrm{C} + y\mathrm{S}\big) \\ \big(\mathbb{i}^{\downuparrows}\big)^*\big(y\mathrm{C} - x\mathrm{S}\big) \\ \big(\mathbb{i}^{\downuparrows}\big)^* z \end{bmatrix}^T \begin{bmatrix} \pm\mathbb{i}^{\updownarrows} t \\ \pm\mathbb{i}^{\downuparrows}\big(x\mathrm{C} + y\mathrm{S}\big) \\ \pm\mathbb{i}^{\downuparrows}\big(y\mathrm{C} - x\mathrm{S}\big) \\ \pm\mathbb{i}^{\downuparrows} z \end{bmatrix} \\
&= \pm|\mathbb{i}^{\updownarrows} t|^2 \pm |\mathbb{i}^{\downuparrows}|^2 \Big[\big(x\mathrm{C} + y\mathrm{S}\big)^2 + \big(y\mathrm{C} - x\mathrm{S}\big)^2 + z^2\Big] \\
&= \pm|\mathbb{i}^{\updownarrows} t|^2 \pm |\mathbb{i}^{\downuparrows}|^2 \big[\big(x^2\mathrm{C}^2 + 2xy\mathrm{CS} + y^2\mathrm{S}^2\big) + \big(y^2\mathrm{C}^2 - 2xy\mathrm{CS} + x^2\mathrm{S}^2\big) + z^2\big] \\
&= \pm|\mathbb{i}^{\updownarrows} t|^2 \pm |\mathbb{i}^{\downuparrows}|^2 \big[x^2\big(\mathrm{C}^2 + \mathrm{S}^2\big) + y^2\big(\mathrm{S}^2 + \mathrm{C}^2\big) + z^2\big] \\
&= \pm|\mathbb{i}^{\updownarrows} t|^2 \pm |\mathbb{i}^{\downuparrows}|^2 \big(x^2 + y^2 + z^2\big)
\end{aligned}$$

Since $|\mathbb{i}^{\updownarrows}|^2 = 1$, the inner product has not been preserved. The spacelike and timelike parts have the same sign, and there is no possibility for a null 4-vector with vanishing Minkowski length (SQUARED?).

Theorem 4.4.22 *The contraction of $x^\mu, x^\nu \in \mathcal{H}$ with the metric is a Lorentz scalar under rotations.*

Proof. Using $x^{\mu'}$ as in Equation (4.2), the contraction with the metric is

$$
\begin{aligned}
\left(x^{\mu'}\right)^2 &= x^{\mu'}\gamma_{\mu\nu}x^{\nu'} \\
&= \begin{bmatrix} \mathbb{i}^{\updownarrows}t \\ \mathbb{i}^{\downuparrows}\left(x\mathrm{C}+y\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}\left(y\mathrm{C}-x\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}z \end{bmatrix}^T \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\updownarrows}t \\ \mathbb{i}^{\downuparrows}\left(x\mathrm{C}+y\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}\left(y\mathrm{C}-x\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}z \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{i}^{\updownarrows}t \\ \mathbb{i}^{\downuparrows}\left(x\mathrm{C}+y\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}\left(y\mathrm{C}-x\mathrm{S}\right) \\ \mathbb{i}^{\downuparrows}z \end{bmatrix}^T \begin{bmatrix} \pm\mathbb{i}^{\updownarrows}t \\ \pm\mathbb{i}^{\downuparrows}\left(x\mathrm{C}+y\mathrm{S}\right) \\ \pm\mathbb{i}^{\downuparrows}\left(y\mathrm{C}-x\mathrm{S}\right) \\ \pm\mathbb{i}^{\downuparrows}z \end{bmatrix} \\
&= \pm\left(\mathbb{i}^{\updownarrows}t\right)^2 \pm \left(\mathbb{i}^{\downuparrows}\right)^2 \left[\left(x\mathrm{C}+y\mathrm{S}\right)^2 + \left(y\mathrm{C}-x\mathrm{S}\right)^2 + z^2\right] \\
&= \pm\left(\mathbb{i}^{\updownarrows}t\right)^2 \pm \left(\mathbb{i}^{\downuparrows}\right)^2 \left[\left(x^2\mathrm{C}^2 + 2xy\mathrm{CS} + y^2\mathrm{S}^2\right) + \left(y^2\mathrm{C}^2 - 2xy\mathrm{CS} + x^2\mathrm{S}^2\right) + z^2\right] \\
&= \pm\left(\mathbb{i}^{\updownarrows}t\right)^2 \pm \left(\mathbb{i}^{\downuparrows}\right)^2 \left[x^2\left(\mathrm{C}^2+\mathrm{S}^2\right) + y^2\left(\mathrm{S}^2+\mathrm{C}^2\right) + z^2\right] \\
&= \pm\left(\mathbb{i}^{\updownarrows}t\right)^2 \pm \left(\mathbb{i}^{\downuparrows}\right)^2\left(x^2+y^2+z^2\right)
\end{aligned}
$$

Since $(\mathbb{i}^{\updownarrows})^2$ and $(\mathbb{i}^{\downuparrows})^2$ are oppositely signed, the contraction with the metric is a Lorentz scalar. The scalar output of the contraction operation agrees with the result obtain in Theorem 4.2.1.

Theorem 4.4.23 *A 4-vector $x^\mu \in \mathbb{R}^4$ is a Lorentz scalar under boosts.*

Proof. We will consider a boost in the z-direction such that the Lorentz transformation may be written

$$
\Lambda^{\mu'}_{\ \nu} = \begin{bmatrix} \cosh\phi & 0 & 0 & -\sinh\phi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\phi & 0 & 0 & \cosh\phi \end{bmatrix} \quad ,
$$

so

$$
x^{\mu'} = \begin{bmatrix} \mathrm{Ch} & 0 & 0 & -\mathrm{Sh} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{Sh} & 0 & 0 & \mathrm{Ch} \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} t\mathrm{Ch} - z\mathrm{Sh} \\ x \\ y \\ -t\mathrm{Sh} + z\mathrm{Ch} \end{bmatrix} \quad .
$$

The contraction with the metric is

$$
\begin{aligned}
\big(x^{\mu'}\big)^2 &= x^{\mu'}\eta_{\mu\nu}x^{\nu'} \\
&= \begin{bmatrix} \mathrm{Ch}t-\mathrm{Sh}z \\ x \\ y \\ -\mathrm{Sh}t+\mathrm{Ch}z \end{bmatrix}^T \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \mp1 & 0 & 0 \\ 0 & 0 & \mp1 & 0 \\ 0 & 0 & 0 & \mp1 \end{bmatrix} \begin{bmatrix} \mathrm{Ch}t-\mathrm{Sh}z \\ x \\ y \\ -\mathrm{Sh}t+\mathrm{Ch}z \end{bmatrix} \\
&= \begin{bmatrix} \mathrm{Ch}t-\mathrm{Sh}z \\ x \\ y \\ -\mathrm{Sh}t+\mathrm{Ch}z \end{bmatrix}^T \begin{bmatrix} \pm\big(\mathrm{Ch}t-\mathrm{Sh}z\big) \\ \mp x \\ \mp y \\ \mp\big(-\mathrm{Sh}t+\mathrm{Ch}z\big) \end{bmatrix} \\
&= \pm\big(\mathrm{Ch}t-\mathrm{Sh}z\big)^2 \mp \Big[x^2+y^2+\big(-\mathrm{Sh}t+\mathrm{Ch}z\big)^2\Big] \\
&= \pm\big(\mathrm{Ch}t-\mathrm{Sh}z\big)^2 \mp \big(-\mathrm{Sh}t+\mathrm{Ch}z\big)^2 \mp \big[x^2+y^2\big] \\
&= \pm\big(t^2\mathrm{Ch}^2-2tz\mathrm{ChSh}+z^2\mathrm{Sh}^2\big) \mp \big(t^2\mathrm{Sh}^2-2tz\mathrm{ChSh}+z^2\mathrm{Ch}^2\big) \mp \big[x^2+y^2\big] \\
&= \pm\big(t^2\mathrm{Ch}^2-2tz\mathrm{ChSh}+z^2\mathrm{Sh}^2-t^2\mathrm{Sh}^2+2tz\mathrm{ChSh}-z^2\mathrm{Ch}^2\big) \mp \big[x^2+y^2\big] \\
&= \pm\big[t^2\big(\mathrm{Ch}^2-\mathrm{Sh}^2\big)+z^2\big(\mathrm{Sh}^2-\mathrm{Ch}^2\big)\big] \mp \big[x^2+y^2\big] \\
&= \pm\big(t^2-z^2\big) \mp \big[x^2+y^2\big] \\
&= \pm t^2 \mp \big[x^2+y^2+z^2\big]
\end{aligned}
$$

A real-valued 4-vector $x^\mu \in \mathbb{R}^2$ is a Lorentz scalar under boosts.

Remark 4.4.24 The structure of Lorentz transformations is different in the cases of real- or complex-valued 4 vectors. The conventions are laid out by Roman as follows [4].

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Definition 4.4.25 Roman has used the convention for strictly imaginary time, but we want to use either time or space. The conventions generalize as

KKKKKKKKKKKKKKKKKKKKkk

Remark 4.4.26 DID I NEED TO USE A DIFFERENT ROTATION MATRIX???

Remark 4.4.27 I need to make some distinctions about the representations/generators of the group when the objects it acts on are different kinds of vectors.

Example 4.4.28 If we use the Lorentz boost defined for $x^\mu \in \mathbb{R}^4$, the result is not a Lorentz scalar. Namely, the representation of the Lorentz group is different for complex-valued 4-vectors.

Given

$$x^{\mu'} = \begin{bmatrix} \mathrm{Ch} & 0 & 0 & -\mathrm{Sh} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\mathrm{Sh} & 0 & 0 & \mathrm{Ch} \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow} t \\ \mathbb{i}^{\downarrow\uparrow} x \\ \mathbb{i}^{\downarrow\uparrow} y \\ \mathbb{i}^{\downarrow\uparrow} z \end{bmatrix} = \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} - \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh} \\ \mathbb{i}^{\downarrow\uparrow} x \\ \mathbb{i}^{\downarrow\uparrow} y \\ -\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} + \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch} \end{bmatrix} ,$$

the contraction with the metric is

$$\begin{aligned}
\left(x^{\mu'}\right)^2 &= x^{\mu'} \gamma_{\mu\nu} x^{\nu'} \\
&= \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} - \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh} \\ \mathbb{i}^{\downarrow\uparrow} x \\ \mathbb{i}^{\downarrow\uparrow} y \\ -\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} + \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch} \end{bmatrix}^T \begin{bmatrix} \pm1 & 0 & 0 & 0 \\ 0 & \pm1 & 0 & 0 \\ 0 & 0 & \pm1 & 0 \\ 0 & 0 & 0 & \pm1 \end{bmatrix} \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} - \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh} \\ \mathbb{i}^{\downarrow\uparrow} x \\ \mathbb{i}^{\downarrow\uparrow} y \\ -\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} + \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch} \end{bmatrix} \\
&= \begin{bmatrix} \mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} - \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh} \\ \mathbb{i}^{\downarrow\uparrow} x \\ \mathbb{i}^{\downarrow\uparrow} y \\ -\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} + \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch} \end{bmatrix}^T \begin{bmatrix} \pm\mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} \mp \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh} \\ \pm\mathbb{i}^{\downarrow\uparrow} x \\ \pm\mathbb{i}^{\downarrow\uparrow} y \\ \mp\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} \pm \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch} \end{bmatrix} \\
&= \big(\mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} - \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh}\big)\big(\pm\mathbb{i}^{\uparrow\downarrow} t\mathrm{Ch} \mp \mathbb{i}^{\downarrow\uparrow} z\mathrm{Sh}\big) \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 x^2 \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 y^2 + \ldots \\
&\qquad \cdots + \big(-\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} + \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch}\big)\big(\mp\mathbb{i}^{\uparrow\downarrow} t\mathrm{Sh} \pm \mathbb{i}^{\downarrow\uparrow} z\mathrm{Ch}\big) \\
&= \pm\big(\mathbb{i}^{\uparrow\downarrow}\big)^2 t^2\mathrm{Ch}^2 \mp 2itz\mathrm{ChSh} \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 z^2\mathrm{Sh}^2 \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2\big(x^2+y^2\big) + \ldots \\
&\qquad \cdots + \Big[\pm\big(\mathbb{i}^{\uparrow\downarrow}\big)^2 t^2\mathrm{Sh}^2 \mp 2itz\mathrm{ChSh} \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 z^2\mathrm{Ch}^2\Big] \\
&= \pm\big(\mathbb{i}^{\uparrow\downarrow}\big)^2 t^2\mathrm{Ch}^2 \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 z^2\mathrm{Sh}^2 \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2\big(x^2+y^2\big) + \ldots \\
&\qquad \cdots + \Big[\pm\big(\mathbb{i}^{\uparrow\downarrow}\big)^2 t^2\mathrm{Sh}^2 \pm \big(\mathbb{i}^{\downarrow\uparrow}\big)^2 z^2\mathrm{Ch}^2\Big]
\end{aligned}$$

Since $\mathbb{i}^{\uparrow\downarrow}\mathbb{i}^{\downarrow\uparrow} = i$, we obtain an i in the cross terms, as should be expected when mixing the time and space parts of $x^\mu \in \mathcal{H}$. The cross terms have the same sign and cannot cancel, so we have proven that we need a different representation of the Lorentz group for $x^\mu \in \mathcal{H}$ than we use when $x^\mu \in \mathbb{R}^4$.

Remark 4.4.29 Compare Jaffe [9] and Steane [10] to Roman [4].

Theorem 4.4.30 *A complex 4-vector $x^\mu \in \mathcal{H}$ is a Lorentz scalar under boosts.*

Proof.

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and the other follows from the transpose operation

$$\big(\Lambda_{\mu\rho}\Lambda_{\mu\sigma}\big)^T = \delta^T_{\rho\sigma} \qquad \Longrightarrow \qquad \Lambda_{\sigma\mu}\Lambda_{\rho\mu} = \delta_{\rho\sigma} \quad .$$

§5 MCM Spinors

§5.1 Intentions

§5.2 The Weyl Equation

§5.3 The Pauli Equation

§5.4 The Dirac Equation

§6 MCM Particles

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