K as a Summation of K Elements

The Waterbed Principle

Let

$$\sum_{i=1}^{K} a_i = K$$

where K is an integer greater than 1. One solution is $a_i = 1$, what can be said about other solutions? In the case that all terms are equal, $a_i = c$, and

$$\sum_{i=1}^{K} a_i = \sum_{i=1}^{K} c = Kc = K$$

$$\longrightarrow c = 1$$

Therefore, if all terms are equal, they all equal 1. If any term does not equal 1, then the terms cannot all be equal.

Assume all terms equal 1 except for one term $a_j \neq 1, 1 \leq j \leq K$. Then

$$\sum_{i=1}^{K} a_i = (K-1)(1) + a_j = K$$

$$\longrightarrow a_j = 1$$

By contradiction, there cannot be exactly one term that is not equal to 1. Therefore, if any term is not equal to 1, at least 2 must not equal 1.

Assume that not all terms equal 1. Then at least 2 terms must not equal 1, and the terms cannot all be equivalent. Therefore, there is a minimum term a_m and maximum term a_M such that

$$a_m \leq a_i \leq a_M$$

$$a_m \neq a_M$$

Then,

$$\begin{aligned}
a_m &\le a_1 \\
2a_m &\le a_1 + a_2 \\
\vdots & \quad\quad \vdots \\
Ka_m &\le \sum_{i=1}^{K} a_i = K \\
a_m &\le 1
\end{aligned}$$

By similar logic, $a_M \ge 1$.

Rearranging the original equation:

$$\begin{aligned}
&\sum_{i=1}^{K} a_i = K \\
&\sum_{i=1}^{K} a_i - K = 0 \\
&\sum_{i=1}^{K} a_i - \sum_{i=1}^{K}(1) = 0 \\
&\sum_{i=1}^{K}(a_i - 1) = 0
\end{aligned}$$

Assume a limiting case $a_m = 1$. Then $a_i \ge 1$, and $(a_i - 1) \ge 0$. Since not all a_i $= 1$, at least two terms must not equal 1. Since all terms $a_i \ge a_m = 1$, at least two terms must be greater than 1. Then

For all terms $\quad (a_i - 1) \ge 0$
For at least two terms $\quad (a_i - 1) > 0$

This implies that $\sum_{i=1}^{K}(a_i - 1) > 0$

By contradiction, $a_m \ne 1$. By similar logic, $a_M \ne 1$.

In summary, if $\sum_{i=1}^{K} a_i = K$ and any term $a_i \ne 1$, then there exist a minimum term $a_m < 1$ and a maximum term $a_M > 1$, and $a_m \le a_i \le a_M$. If a waterbed with an initially flat surface somewhere sinks, then elsewhere must rise.